UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2011 1:30 pm to 3:30 pm

PAPER 10

ANALYSIS OF BOOLEAN FUNCTIONS

Attempt no more than ***TWO*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

This question is about the arithmetic removal lemma. Suppose that $G := \mathbb{F}_2^n$ and $A \subset G$. Then we write $T(A)$ for the proportion of additive triples in A, that is the proportion of pairs $(x, y) \in G^2$ for which $x, y, x + y \in A$, so that

$$T(A) = \int 1_A(x)1_A(y)1_A(x+y)d\mu_G(x)d\mu_G(y).$$

Give, with proof, an example of a set $A \subset G$ with $\mu_G(A) = \Omega(1)$ and $T(A) = 0$.

Prove the arithmetic removal lemma, that is prove the following. Suppose that $A \subset G$ is such that if $A' \subset A$ has $T(A') = 0$ then $\mu_G(A \setminus A') \geqslant \epsilon$. Then $T(A) = \Omega_\epsilon(1)$.

Now write $Q(A)$ for for the proportion of additive quadruples in A, that is the proportion of triples $(x, y, z) \in G^3$ for which $x, y, z, x + y + z \in A$, so that

$$Q(A) = \int 1_A(x)1_A(y)1_A(z)1_A(x+y+z)d\mu_G(x)d\mu_G(y).$$

Show that if $A \subset G$ has $\mu_G(A) \geqslant \epsilon$ then $Q(A) \geqslant \epsilon^4$.

2

This question concerns Boolean influence. Suppose that $G = \{0,1\}^n$ (thought of as a vector space over $\mathbb{F}_2$) and write $(e_i)_i$ for the canonical basis of G so that e_i is 1 in the ith co-ordinate and 0 elsewhere. Given $x \in G$ write x_i for $x \cdot e_i$, and suppose that $\epsilon \in (0,1]$. Define

$$p_\epsilon(x) := \prod_{i=1}^{n} (1 + \epsilon(-1)^{x_i}).$$

Prove Beckner's inequality that

$$\|p_\epsilon * f\|_{L^2(G)} \leqslant \|f\|_{L^{1+\epsilon^2}(G)} \text{ for all } f \in L^{1+\epsilon^2}(G).$$

Hence prove that if $A \subset G$ has density $\alpha > 0$ then

$$\sum_{\gamma:|\gamma|=d} |\widehat{1_A}(\gamma)|^2 \leqslant O(\log 2\alpha^{-1})^d \alpha^2,$$

where $|\gamma|$ is the number of is such that $\gamma(e_i) = -1$.

Finally recall that given a Boolean function $f : G \to \{0,1\}$ the ith influence is defined to be

$$\sigma_i(f) := \int |f_i|^2 d\mu_G(x) \text{ where } f_i(x) = f(x) - f(x + e_i).$$

Prove the KKL theorem, that if $\mathrm{Var}(f) = \Omega(1)$, then there is some i such that

$$\sigma_i(f) = \Omega\left(\frac{\log n}{n}\right).$$

3

This question concerns the Balog-Szemerédi-Gowers lemma.

(a) Suppose that $G := \mathbb{F}_2^n$ and $A \subset G$. We write

$$E(A) := \sum_{x+y=z+w} 1_A(x)1_A(y)1_A(z)1_A(w)$$

for the additive energy of A, and we define the symmetry set of A at threshold η to be

$$\mathrm{Sym}_\eta(A) := \{x \in G : 1_A * 1_A(x) \geqslant \eta\mu_G(A)\}.$$

Suppose that $E(A) \geqslant c|A|^3$ and put $S := \mathrm{Sym}_{c/2}(A)$. Prove that

$$\langle 1_A * 1_A, 1_S\rangle_{L^2(G)} \geqslant c\mu_G(A)^2/2. \tag{1}$$

Let $X_1, \ldots, X_r$ be elements of A chosen uniformly at random and put

$$A' := \{x \in A : x + X_i \in S \text{ for all } i \in \{1, \ldots, r\}\}.$$

Using Hölder's inequality and (1) (or otherwise) show that

$$\mathbb{E}|A'|^2 \geqslant (c/2)^{2r}|A|^2.$$

Now put $B := \{(x, y) \in A'^2 : x + y \notin \mathrm{Sym}_{c^3/8}(S)\}$, and show that

$$\mathbb{E}|B| \leqslant \frac{1}{2^r}\mathbb{E}|A'|^2.$$

By picking r suitably in terms of ϵ show that there are values for the X_is such that

$$|A'| \geqslant c^{O(\log \epsilon^{-1})}|A| \text{ and } |B| \leqslant \epsilon|A'^2|.$$

We have shown that if $E(A) \geqslant c|A|^3$ then there is a set $A' \subset A$ with $|A'| \geqslant c^{O(\log \epsilon^{-1})}|A|$ such that

$$|\{(x, y) \in A'^2 : x + y \in \mathrm{Sym}_{c^3/8}(S)\}| \geqslant (1 - \epsilon)|A'|^2.$$

(b) The above result was the original driving ingredient of the Balog-Szemerédi-Gowers lemma, but now it is more common to use the following result.

Suppose that $E(A) \geqslant c|A|^3$ and $\epsilon \in (0, 1]$ is a parameter. Then there is a subset $A' \subset A$ with $|A'| = \Omega(c|A|)$ such that

$$|\{(x, y) \in A'^2 : x + y \in \mathrm{Sym}_{\epsilon c^2/2}(A)\}| \geqslant (1 - \epsilon)|A'|^2.$$

Assuming this last result prove the Balog-Szemerédi-Gowers lemma that if $E(A) \geqslant c|A|^3$ then there is a subset $A' \subset A$ with $|A'| \geqslant c^{O(1)}|A|$ and $|A' + A'| \leqslant c^{-O(1)}|A'|$.

4

The objective of this question is to prove a slight variant of the Rough Morphism Theorem. Freĭman's theorem, the Balog-Szemerédi-Gowers lemma and Chang's theorem may all be assumed.

Suppose that $S \subset G$ and $\phi : G \to G$ is such that

$$\mu_{G^2}(\{(x,y) : \phi(x+y) = \phi(x) + \phi(y) \text{ and } x, y, x+y \in S\}) \geqslant c$$

for some $c > 0$. Prove that there is a homomorphism $\theta : G \to G$ such that

$$\mu_G(\{x \in S : \phi(x) = \theta(x)\}) \geqslant \exp(-O(c^{-O(1)})).$$

END OF PAPER