UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2011 9:00 am to 12:00 pm

PAPER 1

INTRODUCTION TO LIE ALGEBRAS AND THEIR REPRESENTATIONS

*Attempt **ALL** questions.*

*There are **FIVE** questions in total.*

*Question **THREE** carries the most weight.*

All Lie algebras are over $\mathbb{C}$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

i) Let $\mathfrak{g}$ be an *abelian* Lie algebra. Show that the irreducible representations of $\mathfrak{g}$ are one dimensional, and are parameterized by linear maps $\lambda : \mathfrak{g} \to \mathbb{C}$.

Must every indecomposable representation of $\mathfrak{g}$ be irreducible?

Now let V be an arbitrary finite dimensional representation of $\mathfrak{g}$. Describe a canonical decomposition $V = \oplus_\lambda V^\lambda$ into submodules.

ii) Define what it means for a Lie algebra $\mathfrak{b}$ to be *solvable*.

Describe all *irreducible* representations of $\mathfrak{b}$. State clearly any theorems you use.

2

(i) Let $\mathfrak{g}$ be a Lie algebra, and V a representation of $\mathfrak{g}$. Describe the $\mathfrak{g}$ action on V^*.

Suppose that V is isomorphic to V^*, and V is irreducible. Show the induced bilinear form on V is either symmetric or anti-symmetric.

(ii) Let $\mathfrak{g} = \mathfrak{sl}_2$. Show that *every* finite dimensional representation of $\mathfrak{g}$ is self dual.

For each irreducible finite dimensional representation, determine whether the bilinear form on it is symmetric or antisymmetric.

3

Let

$$\mathfrak{g} = \mathfrak{so}_{2n} = \left\{A \in Mat_{2n} | AJ + JA^T = 0\right\}$$

where $J = \begin{pmatrix} & & & & & 1 \\ & & & & \cdots & \\ & & & 1 & & \\ & & 1 & & & \\ & \cdots & & & & \\ 1 & & & & & \end{pmatrix}$, and let $\mathfrak{t} =$ diagonal matrices in $\mathfrak{g}$.

(i) Decompose $\mathfrak{g}$ as a $\mathfrak{t}$-module, and hence write the roots R for $\mathfrak{g}$. Choose positive roots to be those occurring in upper triangular matrices. Write down the positive roots R^+, the simple roots π, the highest root θ, and the fundamental weights.

Write down ρ.

Draw the Dynkin diagram and label it by simple roots. Draw the extended Dynkin diagram.

(ii) Show that $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$.

(iii) For each root $\alpha \in R$, write the reflection $s_\alpha : \mathfrak{t} \to \mathfrak{t}$ explicitly. Describe the Weyl group W (you do not need to prove your answer).

(iv) Let $V = \mathbb{C}^{2n}$ be the standard representation of $\mathfrak{so}_{2n}$. Draw the crystal of V, and of $V \otimes V$. Write the highest weight of each irreducible summand of $V \otimes V$.

4

(i) State the Weyl character formula, and the Weyl dimension formula briefly defining the notation you use.

(ii) Draw the root system of B_2, and the fundamental weights ω_1, ω_2.

Write down the dimension of the irreducible representation with highest weight $n_1\omega_1 + n_2\omega_2, \quad n_1, n_2 \in \mathbb{N}$.

5

Draw the root system of G_2, and the fundamental weights ω_1, ω_2.

Let ω_1 denote the shorter fundamental weight. [The representation with highest weight ω_2 is the adjoint representation.]

Draw the weights of the representation with highest weight ω_1, and its crystal.

END OF PAPER