UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 2 June, 2010 9:00 am to 11:00 am

PAPER 9

RAMSEY THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

By considering a colouring in which the colour of the edge from i to $i+j$ is always different to the colour of the edge from i to $i+2j$, show that there exists a red-blue colouring of $\mathbb{N}^{(2)}$ in which there is no 3-term arithmetic progression M with $M^{(2)}$ monochromatic.

Now let m be a positive integer. Show that for any red-blue colouring of $\mathbb{N}^{(2)}$ there exists *either* an m-term arithmetic progression M with $M^{(2)}$ blue *or* disjoint m-term arithmetic progressions A and B with every edge from A to B red.

[*Hint: Suppose that every m-term arithmetic progression has at least one of its $\binom{m}{2}$ edges red. This gives a colouring of $\mathbb{N}^2$ with $\binom{m}{2}$ colours, by colouring $(a,d) \in \mathbb{N}^2$ according to which edge of the arithmetic progression with first term a and common difference d is red. Now apply Gallai's theorem.*]

2

State and prove Rado's theorem.

[*You may assume that, for any m, p, c, whenever $\mathbb{N}$ is finitely coloured there is a monochromatic (m,p,c)-set.*]

Deduce that, for any k, whenever $\mathbb{N}$ is finitely coloured there exist $x_1, x_2, \ldots, x_k$ with $FS(x_1, x_2, \ldots, x_k)$ monochromatic.

3

Using Hindman's theorem, show that there exists an ultrafilter on $\mathbb{N}$, each member of which contains a set of the form $FS(x_1, x_2, \ldots)$ (where $x_1, x_2, \ldots \in \mathbb{N}$).

[*Hint: which are the sets that must belong to such an ultrafilter?*]

By considering the set $\{x \in \mathbb{N} : 2^n \leqslant x < 2^{n+1}, \text{ some even } n\}$, or otherwise, show that there is more than one such ultrafilter.

4

What does it mean to say that a subset of $\mathbb{N}^{(\omega)}$ is *Ramsey*? Give an example of a set that is not Ramsey.

Prove that every *-meagre set is *-nowhere-dense. [*You may assume the Galvin-Prikry lemma.*]

Find, with justification, examples of each of the following:

(i) A set that is τ-meagre but not τ-nowhere-dense,

(ii) A countable union of *-closed sets that is not *-closed.

END OF PAPER