UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Wednesday, 2 June, 2010 9:00 am to 11:00 am

## PAPER 9

## RAMSEY THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

By considering a colouring in which the colour of the edge from $i$ to $i+j$ is always different to the colour of the edge from $i$ to $i+2j$, show that there exists a red-blue colouring of $\mathbb{N}^{(2)}$ in which there is no 3-term arithmetic progression $M$ with $M^{(2)}$ monochromatic.

Now let $m$ be a positive integer. Show that for any red-blue colouring of $\mathbb{N}^{(2)}$ there exists *either* an $m$-term arithmetic progression $M$ with $M^{(2)}$ blue *or* disjoint $m$-term arithmetic progressions $A$ and $B$ with every edge from $A$ to $B$ red.

[*Hint: Suppose that every $m$-term arithmetic progression has at least one of its $\binom{m}{2}$ edges red. This gives a colouring of $\mathbb{N}^2$ with $\binom{m}{2}$ colours, by colouring $(a,d) \in \mathbb{N}^2$ according to which edge of the arithmetic progression with first term $a$ and common difference $d$ is red. Now apply Gallai's theorem.*]

**2**

State and prove Rado's theorem.

[*You may assume that, for any $m, p, c$, whenever $\mathbb{N}$ is finitely coloured there is a monochromatic $(m, p, c)$-set.*]

Deduce that, for any $k$, whenever $\mathbb{N}$ is finitely coloured there exist $x_1, x_2, \ldots, x_k$ with $FS(x_1, x_2, \ldots, x_k)$ monochromatic.

**3**

Using Hindman's theorem, show that there exists an ultrafilter on $\mathbb{N}$, each member of which contains a set of the form $FS(x_1, x_2, \ldots)$ (where $x_1, x_2, \ldots \in \mathbb{N}$).

[*Hint: which are the sets that must belong to such an ultrafilter?*]

By considering the set $\{x \in \mathbb{N} : 2^n \leqslant x < 2^{n+1}, \text{some even } n\}$, or otherwise, show that there is more than one such ultrafilter.

**4**

What does it mean to say that a subset of $\mathbb{N}^{(\omega)}$ is *Ramsey*? Give an example of a set that is not Ramsey.

Prove that every *-meagre set is *-nowhere-dense. [*You may assume the Galvin-Prikry lemma.*]

Find, with justification, examples of each of the following:

(i) A set that is $\tau$-meagre but not $\tau$-nowhere-dense,

(ii) A countable union of *-closed sets that is not *-closed.

## END OF PAPER