UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 1 June, 2010 1:30 pm to 4:30 pm

PAPER 8

TOPICS IN FOURIER ANALYSIS AND COMPLEX VARIABLE

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Show that any continuous function of two variables $f : \mathbb{R}^2 \to \mathbb{R}$ can be written in terms of continuous functions of one variable and addition.

2

Let E be a subset of $C(\mathbb{T}^n)$ the space of real valued continuous function on $\mathbb{T}^n$ and let $\epsilon > 0$. Define the Kolmogorov entropy $H(\epsilon, E)$.

Let $B_{n,p}$ be the unit ball in the space of real valued continuously p times differentiable functions $C^p(\mathbb{T}^n)$ with an appropriate norm [$p \geqslant 1$]. Show that there exist non-zero positive constants C_1, C_2 such that

$$C_1 \epsilon^{-n/p} \leqslant H(\epsilon, B_{n,p}) \leqslant C_2 \epsilon^{-n/p} \log \epsilon^{-1}$$

for all $O < \epsilon < 1/2$. What happens if we replace $C^p(\mathbb{T}^n)$ by $C(\mathbb{T}^n)$?

Suppose E is a subset of $C(\mathbb{T}^n)$ with $H(\epsilon, E) \leqslant C_3 \epsilon^{-m/p} \log \epsilon^{-1}$ for some constant C_3 and all $0 < \epsilon < 1/2$. If F is the collection of functions $g \in C(\mathbb{T}^n)$ with

$$g(\mathbf{t}) = f\big(u_1(\mathbf{t}), u_2(\mathbf{t}), \ldots, u_m(\mathbf{t})\big)$$

where $f \in B_{m,p}$ and $u_j \in E$. Show that there is a constant C_4 such that

$$H(\epsilon, F) \leqslant C_4 \epsilon^{-m/p} \log \epsilon^{-1}$$

for all $O < \epsilon < 1/2$.

[*You may use any version of Jackson's theorem that you wish provided it is correctly stated.*]

3

State and prove the Riemann mapping theorem. You may assume the existence of a logarithm function on a simply connected open set not containing zero. You may assume general results from topology and complex variable but not those results whose main purpose in the course is to prove the theorem.

4

(i) State and prove Schwarz's lemma. Find all the comformal maps of the unit disc onto itself.

(ii) Show that if $f : \mathbb{T} \to \mathbb{C}$ is integrable and $\hat{f}(n) = 0$ for $n < 0$ then f cannot have a discontinuity of the first kind. If you use results on conjugate sums of Fourier series you should prove them.

[*Part (ii) carries twice as many marks as part (i).*]

END OF PAPER