UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 8 June, 2010 1:30 pm to 4:30 pm

PAPER 75

TOPOS THEORY

*Attempt no more than **FIVE** questions.*

*There are **TEN** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What is a topos? Show according to your definition that a presheaf category $[\mathbb{C}^{\mathrm{op}}, \mathrm{Set}]$ is a topos.

2

Let $\mathcal{C}$ be a cartesian category. Show that if X in $\mathcal{C}$ has a power object, then so does any subobject of X. Deduce that if $\mathcal{E}$ is a cartesian category with power objects, then so is $\mathcal{E}/A$ for any object A of $\mathcal{E}$.

3

Suppose that $\mathcal{E}$ is a cartesian and cartesian closed category. Let G be a cartesian comonad on $\mathcal{E}$. Show that the category $\mathcal{E}_G$ of G-coalgebras is cartesian and cartesian closed.

4

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism between toposes. Let the map $\overline{\lambda} : f^*(\Omega_{\mathcal{E}}) \to \Omega_{\mathcal{F}}$ classify the subobject $f^*(1) \to f^*(\Omega_{\mathcal{E}})$.

(i) Suppose that $a : X \to \Omega_{\mathcal{E}}$ classifies the subobject $A \rightarrowtail X$. Show that $\overline{\lambda}.f^*(a)$ classifies $f^*A \rightarrowtail f^*X$.

(ii) Show that if the transpose $\lambda : \Omega_{\mathcal{E}} \to f_*(\Omega_{\mathcal{F}})$ is monic then f is a surjection.

(iii) Show conversely that if f is a surjection then λ is monic.

5

(i) Show that a functor $T : \mathbb{C} \to \mathbb{D}$ induces a geometric morphism $[\mathbb{C}^{\mathrm{op}}, \mathrm{Set}] \to [\mathbb{D}^{\mathrm{op}}, \mathrm{Set}]$, whose inverse image is given by composition with T.

(ii) Show that this geometric morphism is a surjection if and only if every object of D is a retract of one in the image of T.

(iii) Show that this geometric morphism is an injection if and only if T is full and faithful.

6

Suppose that $\mathcal{E}$ is a topos, and $\mathcal{L}$ a reflective subcategory with cartesian reflector $L : \mathcal{E} \to \mathcal{L}$.

(i) Explain why $\mathcal{L}$ is cartesian.

(ii) Show that $\mathcal{L}$ is cartesian closed.

(iii) Define the universal closure operation c_L induced by L.

(iv) Let $j : \Omega \to \Omega$ classify the c_L-closure of the generic subobject $1 \to \Omega$. Show that $a : E \to \Omega$ classifies a c_L-closed subobject of E if and only if $j.a = a$.

(v) Explain briefly why Ω_j, the equalizer of j and the identity gives a subobject classifier for $\mathcal{L}$.

7

Suppose that $f : A \to B$ is a map in a category $\mathcal{C}$.

(i) Let $\mathcal{C}$ be a cartesian category. Show that

$$\mathcal{C} \vDash f(x) = f(x') \vdash_{x,x'} x = x'$$

if and only if f is monic.

(ii) Let $\mathcal{C}$ be a regular category. Show that

$$\mathcal{C} \vDash \top \vdash_y \exists x.f(x) = y$$

if and only if f is a cover.

Now suppose that $\mathcal{C}$ is a regular category and $\phi(x, y)$ is a regular formula interpreted in $\mathcal{C}$ with x of sort A and y of sort B. Suppose further that the sequents

$$\phi(x, y) \wedge \phi(x, y') \vdash_{x,y,y'} y = y' \qquad \text{and} \qquad \top \vdash_x \exists y.\phi(x, y)$$

hold in $\mathcal{C}$. Show that $\phi(x, y)$ defines a map $f : A \to B$ in $\mathcal{C}$.

8

What is a coherent category? Suppose that $A \rightarrowtail X$ and $B \rightarrowtail X$ are subobjects of X in a coherent category. Show that $A \vee B$ lies in a pushout diagram of A and B with initial vertex $A \wedge B$.

(You may make use of the internal logic of a coherent category without justification. You need not prove that the subobject lattices in coherent categories are distributive.)

9

What is a classifying topos for a coherent theory? Describe a construction of a classifying topos for a coherent theory, and sketch a proof that your construction does indeed give a classifying topos.

10

Let $\mathbb{P}_1$ be the theory of partial orders with axioms

$$\top \vdash x \leqslant x \ , \ x \leqslant y \wedge y \leqslant z \vdash x \leqslant z \qquad \text{and} \qquad x \leqslant y \wedge y \leqslant x \vdash x = y \,,$$

and $\mathbb{P}_2$ the theory of partial orders with axioms

$$x < y \wedge y < z \vdash x < z \qquad \text{and} \qquad x < x \vdash \bot \,.$$

Identify the classifying toposes $\mathcal{E}_1$ for $\mathbb{P}_1$ and $\mathcal{E}_2$ for $\mathbb{P}_2$ as presheaf toposes. Show that how to interpret $\mathbb{P}_1$ in $\mathbb{P}_2$. Identify the geometric morphism which this gives rise to. Is it a surjection or an injection?

Now let $\mathbb{L}_1$ be $\mathbb{P}_1$ extended by $\top \vdash x \leqslant y \vee y \leqslant x$, and let $\mathbb{L}_2$ be $\mathbb{P}_2$ extended by $\top \vdash x < y \vee x = y \vee y < x$. Are the classifying toposes $\mathcal{F}_1$ for $\mathbb{P}_1$ and $\mathcal{F}_2$ for $\mathbb{P}_2$ presheaf toposes? Does the above geometric morphism restrict to the corresponding subtoposes? Justify your answers.

END OF PAPER