UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 8 June, 2010 1:30 pm to 4:30 pm

PAPER 74

STELLAR AND PLANETARY MAGNETIC FIELDS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A one-dimensional model of a mean field dynamo incorporating fluctuations in α takes the form

$$\frac{\partial A}{\partial t} = -dB \sin(2\pi x/\ell) - r\frac{\partial B}{\partial x} + \frac{\partial^2 A}{\partial x^2} - A\,, \qquad \frac{\partial B}{\partial t} = r\frac{\partial A}{\partial x} + \frac{\partial^2 B}{\partial x^2} - B\,,$$

where r, d are positive constants. The equations are to be solved in $0 < x < \ell$ with the boundary conditions $A = B = 0$ at $x = 0, \ell$.

(i) First set $d = 0$. Show that the equations can be combined into a single equation for the complex function $P = A + iB$, namely

$$\frac{\partial P}{\partial t} = \mathcal{L}_r[P] \equiv ir\frac{\partial P}{\partial x} + \frac{\partial^2 P}{\partial x^2} - P\,.$$

Show that this equation has stationary solutions $P_0(x)$ when $r = r_0 = 2\sqrt{1 + \pi^2/\ell^2}$. By choosing the phase of P_0 appropriately write down solutions of both dipole type (A even, B odd about $x = \ell/2$) and quadrupole type (A odd, B even) for this value of r.

(ii) Now suppose that $0 < d \ll 1$. Expand A, B, r in powers of d so that $A = A_0 + dA_1 + \ldots$, etc. Show that at order d the equations for steady solutions can be written

$$\mathcal{L}_{r_0}[P_1] = -ir_1\frac{\partial P_0}{\partial x} + \frac{1}{2i}(P_0 - P_0^*)\sin(2\pi x/\ell)\,,$$

where $P_0^* = A_0 - iB_0$. Show that $\int_0^\ell P_0^* \mathcal{L}_{r_0}[P_1]\,dx = 0$, and hence obtain an expression for r_1 in terms of integrals. Without evaluating the integrals show that the values for r_1 for dipole and quadrupole solutions are equal and opposite.

2

Write down the equation satisfied by a magnetic field $\mathbf{B}$ in the presence of a conducting fluid whose velocity is of the incompressible stagnation point form $\mathbf{u} = (-\omega r, 0, 2\omega z)$ in cylindrical polar coordinates (r, ϕ, z), with ω a constant. Show that solutions to this equation can be found in the form

$$\mathbf{B} = (0, 0, f(t)\, b(q)\, \cos m\phi)\,, \qquad q = \frac{r}{g(t)}\,.$$

Derive equations governing the form of $f(t), g(t), b(q)$, and give an expression for the time evolution of the total energy per unit length in the z-direction, proportional to $\int_0^\infty f^2 b^2 r\, dr$. Solve for $b(q)$ in the special case $m = 0$, for a field vanishing as $r \to \infty$. It is not required to solve the equation for $b(q)$ when $m \neq 0$.

The total magnetic flux for this solution is zero if $m \neq 0$. In what sense, then, should this flow be considered a dynamo?

[*The Laplacian operator in cylindrical polar coordinates has the form*

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\,.\Bigg]$$

3

An axisymmetric time-independent poloidal field $\mathbf{B}_p$ is prescribed within a conducting fluid sphere of radius a and uniform magnetic diffusivity η and density ρ, surrounded by insulator and rotating at angular velocity $\Omega\hat{\boldsymbol{z}}$ in cylindrical polar coordinates (r, ϕ, z). The flow in the sphere is also axisymmetric and incompressible, with kinematic viscosity ν. The azimuthal flow is asumed to be almost geostrophic, so that there is a time dependent azimuthal flow $U(r,t)\hat{\boldsymbol{\phi}}$ and also a poloidal flow $\mathbf{u}_p(\mathbf{x}, t)$.

Give conditions on $\nu, \eta, |\mathbf{B}_p|, |\mathbf{u}_p|$ etc., that allow the azimuthal components of the full momentum and induction equations to be approximated by the reduced forms

$$\frac{\partial U}{\partial t} + 2\Omega u_r \;=\; (\mu_0\, i\rho)^{-1}\, r^{-1}\nabla\cdot(r\mathbf{B}_p B_\phi)\,, \qquad \frac{\partial B_\phi}{\partial t} \;=\; r\mathbf{B}_p\cdot\nabla(r^{-1}U)\,.$$

Thus, defining $z_b(r) = \sqrt{a^2 - r^2}$, derive the coupled system

$$\frac{\partial U}{\partial t} \;=\; \frac{1}{2z_b\,\mu_0\,\rho r^2}\,\frac{\partial(r^2\mathcal{T})}{\partial r}\,, \qquad \frac{\partial \mathcal{T}}{\partial t} \;=\; H(r)\,r\,\frac{\partial(r^{-1}U)}{\partial r}\,,$$

where

$$\mathcal{T}(r,t) \;=\; \int_{-z_b}^{z_b} B_r\,B_\phi\,dz\,, \qquad H(r) \;=\; \int_{-z_b}^{z_b} B_r^2\,dz\,.$$

Show that this system (which describes 'torsional waves') conserves 'energy' in the sense that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\int_0^a (2z_b\,\mu_0\,\rho U^2 + H^{-1}\mathcal{T}^2)\,r\,dr\right) = 0\,.$$

Give a rough estimate of H for the geomagnetic field and thus give an order of magnitude estimate of the frequency of such torsional oscillations in the Earth's core.

4

Write an essay on anti-dynamo theorems. Your essay should include outline derivations of at least two of the major results (Backus' condition, Cowling's Theorem, the toroidal theorem) together with an overview of other results.

END OF PAPER