UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 4 June, 2010 9:00 am to 12:00 pm

PAPER 72

SOLIDIFICATION OF FLUIDS

A distinction can be gained from substantially complete answers to ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

Candidates may bring into the examination any hand-written notes and materials that were distributed during the course.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a spherical cavity of radius R in a porous medium of permeability Π. The cavity and the pores of the surrounding medium are filled with water and the whole system is initially at a uniform temperature $T_\infty < T_m = 0°\mathrm{C}$. Ice nucleates at the centre of the cavity and grows as a spherical solid of radius $a(t)$. Assume that the pressure inside the cavity is uniform and that flow in the porous medium is governed by Darcy's equation.

Show that the pressure in the cavity is

$$p = p_m + \frac{\mu}{\Pi}\left(1 - \frac{\rho_s}{\rho}\right)\frac{a^2\dot{a}}{R},$$

where p_m is the far-field pressure in the porous medium, equal to atmospheric pressure, ρ and μ are the density and dynamic viscosity of water respectively, and ρ_s is the density of ice.

Determine the temperature field, ignoring the Gibbs–Thomson effect and making the quasi-stationary approximation. Hence show that

$$\left(x + \frac{x^2}{K}\right)\dot{x} = 1, \qquad \text{and} \qquad p - p_m = \frac{x}{x+K}\,p^*,$$

where

$$x(\tau) = a(t)/R, \qquad t = \frac{R^2\rho_s L}{k(T_m - T_\infty)}\,\tau,$$

$$K = \frac{1}{(1/\rho_s - 1/\rho)^2}\,\frac{L^2}{kT_m}\,\frac{\Pi}{\mu}, \qquad p^* = \rho_s L\,\frac{T_m - T_\infty}{T_m}\left(1 - \frac{\rho_s}{\rho}\right)^{-1},$$

k is the thermal conductivity of water and L is the latent heat of fusion. Interpret these results in the limits $K \gg 1$ and $K \ll 1$, giving the leading-order expressions for $p(x)$ and $x(t)$ in each case.

2

The rapid solidification of a pure, supercooled melt can be unstable to morphological instability. When the Stefan number $\mathcal{S} < 1$, the kinetic rates of attachment become important. Consider a linear law relating the growth rate $\dot{a}$ with the interfacial supercooling and write down the interfacial temperature including the effect of curvature for a planar solid-liquid interface growing into a supercooled melt. Find the steady state thermal field and interfacial temperature. Show that for $\mathcal{S} < 1$ the growth rate is constant, $\dot{a} = V$, and is only a function of the Stefan number and the kinetic coefficient.

Next, examine the stability of the solid-liquid interface and show that the growth rate of perturbations σ is given by

$$\sigma \left(1 + \frac{\mathcal{S}}{p}\right) = \mathcal{S}V - \Gamma\alpha^2 - \mathcal{S}V^2/p$$

where V is the non-dimensional interface velocity, Γ is the non-dimensional surface energy density, α is the non-dimensional wavenumber of perturbations and

$$p = \frac{1}{2}\left[V + \sqrt{V^2 + 4(\sigma + \alpha^2)}\right] .$$

Discuss the morphological stability of the interface for $\mathcal{S} = 0$ and $\mathcal{S} = 1$. For $0 < \mathcal{S} < 1$ find the growth rate $\sigma = \sigma(\alpha)$ in the small and large α limits. Assume that $\sigma = O(\alpha^2)$ as $\alpha \to 0$ and show that

$$\sigma \sim \left(\frac{\mathcal{S}}{V} - \Gamma\right)\alpha^2 \qquad \text{as} \quad \alpha \to 0 .$$

Find an expression for σ for $\alpha \gg 1$. Hence sketch $\sigma(\alpha)$ for a few values of Γ and give approximate conditions for morphological instability.

3

Consider the ablation of the vertical terminus of a glacier in a warm ocean, driven by the flux of heat and solute at the ice–ocean interface. Write down expressions for conservation of mass and momentum in a steady turbulent plume of width b and mean vertical velocity w whose rise is driven principally by contrasts in salinity, neglecting the effective wall shear stress. By considering conservation of heat (enthalpy) and solute within a horizontal slab spanning the plume, derive the expressions

$$\frac{d}{dz}\left[bw(T - T_\infty)\right] = \dot{a}\left[\frac{\mathcal{L}}{c_p} + T_\infty - T_s\right],$$

$$\frac{d}{dz}\left[bw(C - C_\infty)\right] = \dot{a}(C_\infty - C_s)\,,$$

where T and C are the mean temperature and salinity within the plume, T_∞ and C_∞ are the temperature and salinity of the far-field ocean, $\dot{a}$ is the ablation rate of the glacial terminus (defined as positive for growth and negative for ablation), C_s is the solute concentration within the ice, T_s is the temperature of the ice and $\mathcal{L}$ and c_p are the latent heat of fusion and the specific heat capacity respectively.

Examine the large Stefan number limit, $\mathcal{L}/c_p \gg T_\infty - T_s$. At the ice-ocean interface the temperature T_i and salinity C_i are in thermodynamic equilibrium. Assume that heat and solute fluxes from the plume to the ice can be approximated by diffusive fluxes across a laminar boundary layer of constant width δ. For simplicity neglect diffusion in the solid, and take the limits $T_\infty - T_i \gg T_\infty - T$ and $C_\infty - C_i \gg C_\infty - C$.

Find power law solutions of the plume equations and determine the interfacial temperature and composition as a function of the compositional Peclet number $Pe_c \equiv -\dot{a}\delta/D$. Show that the rate of ablation is given by

$$\frac{\mathcal{L}}{c_p}\,\frac{D}{\kappa}\,Pe_c = T_\infty - T_m + \Gamma\,\frac{(C_\infty - C_s)}{1 + Pe_c}\,,$$

where the liquidus is approximated by the linear relationship $T_L(C) = T_m - \Gamma(C - C_s)$, D is the diffusivity of solute and κ is the thermal diffusivity. Sketch a typical compositional and thermal profile from the interface, through the laminar boundary layer and across the plume to the far-field. Examine the limits $Pe_c \gg 1$ and $Pe_c \ll 1$ discussing the dominant mechanism driving ablation of the glacier in each limit (melting/dissolution).

4

An aqueous salt solution of concentration C_0, liquidus $T = -mC$ (m constant), segregation coefficient $k_D = 0$ and diffusivity D is pulled downwards at constant speed V through a fixed temperature field $T = T_E + Gz$, where T_E is the eutectic temperature, z measures distance upwards and G is the constant temperature gradient.

Explain why there can be no steady state with completely solid ice growing from the solution. Sketch the temperature and liquidus temperature fields in physical space and the trajectory $(C(z), T(z))$ in the phase diagram when there is a steady, equilibrium mushy layer. Using the marginal equilibrium hypothesis and including diffusion of salt in the mushy layer, show that the constant thickness of the mushy layer

$$h_0 = \frac{-mC_0 - T_E}{G} - \frac{D}{V}$$

and the solid fraction distribution

$$\phi = \frac{h_0 - z}{mC_0/G + h_0 - z} \qquad (0 < z < h_0)\,.$$

At some time $t = 0$, after the steady mushy layer has been established, the system is brought to rest ($V = 0$) in the same fixed temperature field, and an impermeable membrane is inserted at $z = h_0$. Show that during the subsequent evolution of the mushy layer, the solid fraction

$$\phi(z,t) = f(\eta) \qquad \text{where} \qquad \eta = z^2 + (2T_E/G)z + 2Dt$$

and that, therefore, the position $h(t)$ of the mush–liquid interface is given by

$$\left(h + \frac{T_E}{G}\right)^2 = \left(h_0 + \frac{T_E}{G}\right)^2 - 2Dt\,.$$

Use physical reasoning to sketch the solid-fraction distribution at time $t = 0$, at an intermediate time later, and in the final steady state. Characterize the final state as far as you can.

END OF PAPER