UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Thursday, 27 May, 2010 1:30 pm to 4:30 pm

## PAPER 71

## GEOPHYSICAL AND ENVIRONMENTAL FLUID DYNAMICS

*You may attempt **ALL** questions,*
*although full marks can be achieved by good answers to **THREE** questions.*

*Complete answers are preferred to fragments.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Consider a channel of uniform width inclined at a constant angle $\theta$ to the horizontal. Suppose there is a thin turbulent layer of fluid of density $\rho$ and thickness $h$ flowing in the positive $x$ direction along the channel. Here $x$ is taken as parallel with the channel in down-slope direction and $z$ is perpendicular to the channel. The fluid above the thin layer is at rest with a constant density $\rho_0$.

(a) Explain the Boussinesq approximation and how the thinness of the fluid layer can be used to simplify the form of $u$. Relate the 'shallow water' approximation for a gravity current to the 'top-hat' approximation for a turbulent plume.

(b) The drag between the fluid layer and the channel is negligible, but the fluid within the layer is turbulent resulting in 'Batchelor entrainment' described by the entrainment velocity $w_e = \alpha\,|u|$, where $\alpha$ is assumed constant. Derive a suitable system of equations, using the top-hat distribution, and show that the system is hyperbolic provided $\theta \neq \pi/2$.

(c) Suppose the flow is driven by a steady line source (spanning the width of the channel) providing constant buoyancy flux $F_0$ (per unit length) at $x = 0$ with negligible momentum or volume flux. By searching for a power-law solution, determine how the thickness, velocity and density of the thin layer vary along its length. Over what range of angles is this solution valid?

**2**

Consider a two-dimensional stably stratified atmosphere with buoyancy frequency $N(z)$ and mean wind field $\mathbf{U} = (U(z), 0)$ flowing across an extensive range of sinusoidal hills of height $h = h_0 \sin(2\pi x/L)$. Here, $x$ is horizontal in the down-wind direction and $z$ is vertically upwards.

(a) For a uniform wind $U(z) = U_0$ and stratification $N(z) = N_0$, sketch the linear wave field generated by the hills. Label your sketch, showing the directions of the phase and group velocities in the frame of reference of the hills. On a separate sketch, show the directions of the phase and group velocities in a frame of reference moving with the mean wind speed. State any restrictions on $L$, $U_0$ or $h_0$ in this analysis.

(b) State the 'WKB approximation' as it relates to steady internal waves in a stratified shear flow. For a uniform velocity $U(z) = U_0$, sketch the wave field in the frame of the hills when (i) $N = N_0(1 + z/H)$ and (ii) $N = N_0(1 - z/H)$. Identify the wave crests and the direction of energy propagation. You need not compute the actual path taken, but you should comment on any particular features expected.

(c) By linearising the equations of motion in the $x - z$ plane about the mean wind profile $\mathbf{U} = (U(z), 0)$, show that under the WKB approximation

$$m^2 = \left( \frac{N^2}{k^2 U^2} - \frac{U''}{k^2 U} - 1 \right) k^2 ,$$

where $\mathbf{k} = (k, m)$ is the wavenumber vector. What restrictions are placed on $m$ by causality if there are no downward propagating waves? Assuming $U''$ is negligible and $N(z) = N_0$, identify the conditions under which critical layer absorption and critical layer reflection can take place, and describe these phenomena. For the specific profiles $U = U_0(1 + z/H)$ and $U = U_0(1 - z/H)$, identify whether absorption or reflection occur and determine the corresponding critical height $z = \zeta$. Express this height in terms of the topographic length scale $L$ rather than wavenumber components.

(d) Determine the path taken by a packet of wave energy released at $(x, z) = (0, 0)$ when $N(z) = N_0$ and $U(z) = U_0(1 + z/H)$.

**3**

Consider a very long lake of maximum depth $H$ with a uniform parabolic cross-section described by width $b(x, z) = z^{1/2}$ for $z \geqslant 0$.

(a) Derive the shallow water equations for this geometry and show that the surface wave speed is given by

$$c = \sqrt{\frac{2}{3}\, gh}\,,$$

where the water surface is at height $z = h$ and $g$ is gravity. Determine the characteristics of these equations and the evolution equation(s) along these characteristics.

(b) A dam is constructed across the middle of the lake at $x = 0$. A large quantity of sediment is raised from the bottom of the lake during the construction of the dam. At $t = 0$ this sediment occupies a region of depth $h_0 \ll H$ and length $L_0$. The sediment has a density $\rho_p$, settling velocity $W_s$ and initial concentration $\phi_0 \ll 1$. The density of the fresh lake water is $\rho_0 < \rho_p$ so that for $t > 0$ this sediment-laden layer flows as a gravity current along the bottom of the lake. Give an expression for the reduced gravity $g'$ and hence the speed of long waves on the interface between the sediment-laden layer and the fresh lake water. Describe a suitable boundary condition for the front of the current. By assuming the volume of the current is preserved, derive an integral model to determine $L_\infty$, the maximum extent of the current. You may assume the particles still in suspension remain well-mixed throughout the current.

(c) After the dam is completed, the water is drained from the right-hand half of the lake. Sometime later, the dam fails catastrophically and water floods back into the right-hand half of the lake. Describe the initial development of the flow, illustrating key features on an $x - t$ diagram and commenting on the applicability of the shallow water equations. Give an expression for the depth of the flow.

(d) A hydraulic bore forms to the right of the failed dam as the drained half of the lake fills following the failure of the dam. Derive suitable 'jump conditions' relating the flow on either side of the bore to the speed of the bore. Describe, in terms of the characteristics of the flow, the conditions necessary for the bore to be stationary.

**4**

(a) Describe the Stokes drift mechanism for linear surface waves and show that in deep water the drift velocity is given by

$$u_s = \eta_0^2 \, k \, \omega \, e^{2kz} \, ,$$

where the elevation of the surface is $\eta(x,t) = \eta_0 \cos(kx - \omega t)$ with $k$ the wavenumber, $2\pi/\omega$ the wave period and $z$ is directed upwards from the undisturbed surface at $z = 0$.

(b) The dispersion relation for inviscid two-dimensional linear plane internal gravity waves of period $2\pi/\omega$ propagating on a stationary fluid can be expressed as

$$\frac{\omega}{N} = \cos\theta \, ,$$

where $N$ is the buoyancy frequency and $\theta$ is the angle between the wavenumber vector $\mathbf{k} = (k, m)$ and the horizontal $x$ axis. What happens if $\omega > N$? Give expressions for the phase velocity $\mathbf{c}_p$ and group velocity $\mathbf{c}_g$ in terms of $N$, $|\mathbf{k}|$ and $\theta$. Explain why there can be no Stokes drift mechanism for inviscid linear internal gravity waves described by a single wavenumber vector $\mathbf{k}$.

(c) Suppose a steady periodic disturbance at the lower boundary of a semi-infinite fluid with constant $N$ induces a vertical velocity $w(x, z = 0) = 2W \cos(\kappa x) \cos(\omega t)$. This disturbance generates two sets of linear waves described by wavenumber vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ such that $|\mathbf{k}_1| = |\mathbf{k}_2|$ and $\mathbf{k}_1 \cdot \mathbf{k}_2 = 0$. Determine and describe the structure of the resulting wave field. Analyse this wave field to determine the associated Stokes drift. Is this drift physically reasonable?

## END OF PAPER