UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 28 May, 2010 9:00 am to 11:00 am

PAPER 70

WAVE SCATTERING IN INHOMOGENOUS MEDIA

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Describe the main steps of the Wiener-Hopf technique for solving boundary value problems with boundary conditions along an infinite line that have a discontinuity, and state - without proof - the main theorems on which it relies.

(b) Consider the 2-dimensional problem of a plane wave

$$\phi_i \;=\; e^{ikx\,\cos\theta\,-\,iky\,\sin\theta}$$

incident with angle θ to the horizontal x upon an infinite plane at $y = 0$. The plane is characterized by Dirichlet boundary conditions for $x < 0$, and by Neumann boundary conditions for $x > 0$:

$$\phi_t(x,0) \;=\; 0 \qquad\qquad x \;<\; 0\,,$$

$$\frac{\partial\phi_t(x,0)}{\partial y} \;=\; 0 \qquad\qquad x \;>\; 0\,,$$

where ϕ_t is the total field, which may be written as $\phi_t = \phi_i + \phi_r + \phi_d$, where ϕ_i is the incident field, the reflected field ϕ_r is defined as

$$\phi_r \;=\; -e^{ikx\,\cos\theta\,+\,iky\,\sin\theta}$$

and ϕ_d is the field diffracted by the discontinuity. At the discontinuity the field remains bounded, and we have

$$\frac{\partial\phi_t(x,0)}{\partial y} \;\sim\; x^{-1/2} \qquad \text{as } x \;\to\; 0^+\,.$$

Use the Wiener-Hopf technique to find an expression for the diffracted field ϕ_d, hence the total field. You need not calculate the final integral, but should explain how to choose the integration path needed for its calculation.

[Hint: you may find it useful to solve this as a diffraction problem for ϕ_d, by first finding boundary conditions for ϕ_d on the plane $y = 0$.]

2

A time-harmonic scalar wave $\psi e^{-i\omega t}$ is a superposition of plane waves propagating with wavenumber k at small angles to the horizontal x in a 3-dimensional medium (x, y, z). The medium is characterised by refractive index $n(x, y, z)$, defined as

$$n(x, y, z) \;=\; 1 + \mu W(x, y, z)$$

where μ is a constant and $W(x, y, z)$ is the random part, which is normally distributed and statistically stationary, and has been normalised so that $\langle W \rangle = 0$ and $\langle W^2 \rangle = 1$. The slowly-varying part of ψ, $E(x, y, z) = \psi(x, y, z)\, e^{-ikx}$, obeys the parabolic equation.

(a) Derive an equation of propagation for the first moment of the field, $\langle E(x, y, z) \rangle$, and write the solution $\langle E(x, y, z) \rangle$ at a generic point x in the medium.

(b) Assume now that the medium is isotropic, and δ-correlated in the direction of propagation x:

$$\langle W(x,\, y_1,\, z_1) W(x,\, y_2,\, z_2) \rangle \;=\; \delta(x_1 - x_2)\, A(\eta, \zeta)\,,$$

where A is a differentiable function of the distances $\eta = y_1 - y_2$ and $\zeta = z_1 - z_2$.

Given the definition of the power spectrum as the Fourier Transform of an autocorrelation function, express the solution $\langle E(x, y, z) \rangle$ in terms of the power spectrum of the refractive index of the medium.

[*Use the following result:*
If $f(\eta, \zeta)$ is isotropic, so $f(r, \theta) = f(r)$ in polar coordinates (r, θ), and

$$F(\nu_\eta,\, \nu_\zeta) \;=\; \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\eta, \zeta)\, e^{-i(\nu_\eta \eta + \nu_\zeta \zeta)}\, d\eta\, d\zeta\,,$$

then the following applies:

$$F(\nu_\eta,\, \nu_\zeta) \;=\; F(\nu) \;=\; \int_0^{\infty} f(r)\, J_0(\nu r)\, r\, dr\,, \qquad \textit{where} \quad \nu \;=\; |(\nu_\eta,\, \nu_\zeta)|\,,$$

and $J_0(\nu r)$ is a Bessel function, together with the inverse transform

$$f(\nu) \;=\; \int_0^{\infty} F(\nu)\, J_0(\nu r)\, \nu\, d\nu\,.$$

Use also $J_0(0) = 1$.]

(c) Write now an explicit expression for the above solution in the case when the medium has power spectrum given by:

$$S(\nu) \;=\; \mu^2 L^3 e^{-(\nu L)^2/4}\,,$$

where L is the correlation length of the medium.

3

Consider the 2-dimensional problem of a time-harmonic, monochromatic acoustic plane wave incident upon a perfectly reflecting randomly rough surface defined by $z = h(x)$, with mean $\langle h(x) \rangle = 0$ and r.m.s. height σ.

(a) Assuming that the variation in surface height is small, $|kh(x)| \ll 1$, use first order perturbation theory to derive an expression for the scattered field $\psi_s(x, z)$ in the case of Dirichlet boundary conditions: $\psi = 0$ at the surface.

(b) In the case when the surface is statistically stationary, derive an expression for the mean scattered intensity in the far field in terms of the power spectrum of the surface.

(c) Consider now the 3-dimensional problem of an incident time-harmonic, monochromatic, linearly polarized electromagnetic wave with wave vector $\mathbf{k}$ in the (x, z) plane, incident upon a perfectly conducting 2-dimensional surface which has a statistically random profile in x, but is constant in y (i.e. a 'corrugated' surface with $h(x, y) = h(x, y')\ \forall\, y, y'$).

Show that for a TE wave (where the electric field is perpendicular to the x, z plane and parallel to the x, y plane), this problem can be reduced to the 2-dimensional problem considered in (a).

Does this also work for a TM wave (where the magnetic field is perpendicular to the (x, z) plane and parallel to the (x, y) plane)? Give reasons for your answer.

4

(a) Consider a scalar wave $\psi_i(\mathbf{r})$ incident upon an inhomogeneity $V(\mathbf{r})$, with known space-dependant refractive index $n(\mathbf{r}) = 1 + n_\delta(\mathbf{r})$ and embedded in free space.

Express the scattered field $\psi_s(\mathbf{r})$ as an infinite series using the Born approximation.

Comment on the physical significance of the successive terms in the series, and give at least one condition for the validity of the first Born approximation.

(b) Consider now the inverse problem of finding the unknown refractive index $n(\mathbf{r})$ of an extended, finite inhomogeneity embedded in free space, given a known incident field $\psi_i(\mathbf{r})$ and a known, measured scattered field $\psi_s(\mathbf{r})$.

Write a solution of this inverse problem in the first Born approximation, using a formal representation with operators. This problem is ill-posed. State how it can be regularised using Tikhonov reguarisation.

(c) Given a well-posed direct problem $Ax = y$, where A is a bounded operator from a normed space X into a normed space Y, $A : X \mapsto Y$, and y is the unknown, the inverse problem of finding x given A and y can be recast into finding x that minimises $\| Ax - y \|^2$ and is equivalent to solving

$$x = (A^*A)^{-1}A^*y \quad , \tag{1}$$

where $A^* : Y \mapsto X$ is the adjoint of A. Related to A, there exists a 'singular value system' $(\sigma_i; \mathbf{u}_i; \mathbf{v}_i)$, where σ_i^2 are the eigenvalues of A^*A, u_i are the corresponding set of orthonormal eigenvectors of A^*A, and v_i are the complete set of orthonormal eigenvectors of AA^*, so the following relations hold:

$$\begin{aligned} Au_i &= \sigma_i\, v_i \\ A^*v_i &= \sigma_i\, u_i \\ Ax &= \textstyle\sum_i \sigma_i (x, u_i)\, v_i \\ A^*y &= \textstyle\sum_i \sigma_i (y, v_i)\, u_i \\ A^*Au_i &= \sigma_i^2\, u_i \,. \end{aligned}$$

Here (a, b) denotes the inner product, and note that if there are infinitely many singular values then the sums above are infinite and $\lim_{i\to\infty} \sigma_i = 0$.

(i) By using the properties above, or otherwise, show that the inverse problem (1) is ill-posed and, assuming the known measured field y_δ is given with a small error $\delta > 0$ in the direction of an eigenvector v_j : $y_\delta = y + \delta v_j$ (so $\| y_\delta - y \| = \delta$), relate to δ the error $\| x_\delta - x \|$ derived by using y_δ in (1).

(ii) Using Tikhonov regularisation the problem (1) can be made well posed. Show this for the case where we can use the following approximate formula for the inverse of the sum of operators $(\alpha\mathbb{I} + AA^*)$, where $\mathbb{I}$ is the identity:

$$(\alpha\mathbb{I} + AA^*)^{-1} = \alpha^{-1}\mathbb{I} - \alpha^{-2}A(\mathbb{I} - \alpha^{-1}A^*A)A^* \,.$$

END OF PAPER