UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 4 June, 2010 1:30 pm to 4:30 pm

PAPER 7

ATIYAH SINGER INDEX THEOREM

*Attempt question **ONE** and*
*no more than **TWO** other questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Write an essay outlining the supersymmetric proof of the Atiyah–Singer index theorem for twisted Dirac operators on compact spin manifolds.

2

(a) Prove that $H_s(\mathbb{T}^n) \subset C(\mathbb{T}^n)$ for $s > n/2$.

(b) Prove the double commutant theorem for *–algebras on a finite–dimensional inner product space.

(c) Let $f : (a,b) \to M_n(\mathbb{R})$ be a differentiable function. Establish a formula for $\dfrac{d}{dt}\, e^{f(t)}$ in terms of df/dt.

(d) If ∇_X is the Riemannian connection on forms on a compact Riemannian manifold, prove that the exterior derivative is given by $d = \sum_i e(\omega_i)\nabla_{X_i}$ where (X_i) is locally a basis of vector fields and (ω_i) is the dual basis of 1–forms.

3

(a) Prove that a bounded operator on a Hilbert space is Fredholm of index 0 if and only if it is the sum of an invertible operator and a compact operator.

(b) Let V be a finite–dimensional real inner product space with orthonormal basis (e_i). Let $C : V \to M_n(\mathbb{C})$ be a real linear map such that $C(v)^* = C(v)$ and $C(a)\,C(b) + C(b)\,C(a) = 2(a,b)I$ for $a, b \in V$. If T is a skew–adjoint operator on V, prove that $\pi(T) = \frac{1}{4}\sum C(Te_i)\,C(e_i)$ satisfies $\pi(T)^* = -\pi(T)$ and $[\pi(T), C(v)] = C(Tv)$ for all $v \in V$.

(c) Let M be compact manifold and $p \in C^\infty(X, M_n(\mathbb{C}))$ a self–adjoint projection. Prove that $pdp = dp(1-p)$. Deduce that, if $\omega = p(dp)^2$, then $\operatorname{tr}\omega^k$ and $\det f(\omega)$ are closed if $f(t)$ is a polynomial with $f(0) = 1$.

(d) Let D be the Dirac operator acting on a Clifford bundle $E = E_+ \oplus E_-$ over a compact Riemannian manifold. Let $D_\pm = D|_{E_\pm}$. Assuming any spectral properties that you may require, prove that

$$\operatorname{ind} D_+ \;=\; \operatorname{Tr}(e^{D_-D_+t}) - \operatorname{Tr}(e^{D_+D_-t}) .$$

4

(a) Let $L = \Delta + V(x)$ on $T = \mathbb{T}^n$ where $\Delta = -\sum \partial_{x_i}^2$ and $V \in C^\infty(T)$. Suppose that $Lf = u$ with $f \in H_{-\infty}(T)$ and $u \in C^\infty(T)$. Prove that f lies in $C^\infty(T)$.

(b) Let $\mathcal{F}$ be the space of holomorphic functions f on $\mathbb{C}$ such that

$$\pi^{-1} \int |f(z)|^2 \, e^{-|z|^2} \, dx \, dy \; < \; \infty$$

with inner product

$$(f, g) \; = \; \pi^{-1} \int f(z) \, \overline{g(z)} e^{-|z|^2} \, dx \, dy \; .$$

Prove that $\mathcal{F}$ is a Hilbert space with orthonormal basis $e_n(z) = z^n/\sqrt{n!}$ $(n \geqslant 0)$.

(c) Define the Hermite functions $H_n(x)$. Prove that they are eigenfunctions of the harmonic oscillator $-d^2/dx^2 + x^2$. Show that

$$\sum_{n \geqslant 0} H_n(x) \, e_n(z) \; = \; \pi^{-1/4} \, \exp(-x^2/2 + 2xz/\sqrt{2} - z^2/2)$$

and hence or otherwise deduce that for $t \in (0,1)$

$$\sum_{n \geqslant 0} t^n \, H_n(x) \, H_n(y) \; = \; \frac{1}{\sqrt{\pi(1-t^2)}} \; \exp \frac{4\,xyt - (1+t^2)(x^2+y^2)}{2(1-t^2)} \; .$$

(d) State and prove the Hodge theorem for the Laplacian acting on forms on a compact oriented Riemannian manifold. [*You may assume any properties of Sobolev spaces that you require.*]

5

(a) State and prove Gauss' lemma.

(b) Let $g_{ij}(x)$ be a smooth metric on the ball $B = \{x \in \mathbb{R}^n : \|x\| < r\}$ in normal coordinates. Let $g(x)^{-1} = (g^{ij}(x))$ and let $L = -\sum_{ij} \partial_{x_j} g^{ij} \partial_{x_i}$. Prove that there is a unique formal power series in t, $G(x,t) = 1 + \sum_{n \geqslant 1} B_n(x) t^n$, with B_n smooth, such that

$$F(x,t) = (4\pi t)^{-n/2} e^{-\|x\|^2/4t} G(x,t)$$

satisfies $\partial_t F + LF = 0$.

(c) Define the Laplacian Δ acting on functions on an n–dimensional compact Riemannian manifold M. Assuming any facts about Sobolev spaces that you may require, prove that the eigenvalues $0 = \lambda_0 \leqslant \lambda_1 \leqslant \cdots$ of Δ satisfy $\lambda_k \geqslant Ak^{2/n}$ for some constant $A > 0$.

(d) Show that if Δ is the Laplacian acting on functions on a compact Riemannian manifold M, then $e^{-t\Delta}$ $(t > 0)$ is a trace–class operator on $L^2(M)$ given by a smooth kernel $K_t(x,y)$. Establish a formula for $\mathrm{Tr}\, e^{-\Delta t}$ in terms of $K_t(x,y)$.

END OF PAPER