UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Monday, 7 June, 2010 1:30 pm to 4:30 pm

## PAPER 68

## PERTURBATION AND STABILITY METHODS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

(a) Obtain two terms of an asymptotic expansion for each root of the equation

$$x^2 e^{-x} = \epsilon$$

for $\epsilon > 0$ and $\epsilon \to 0$.

(b) State Watson's lemma and sketch a proof of it.

Suppose that

$$f(\lambda) = \int_a^b e^{\lambda\phi(x)} g(x)\,\mathrm{d}x .$$

If $\phi$ is monotonic decreasing in $[a,b]$ and $\phi'(a) \neq 0$ and $g(a) \neq 0$, find two terms of an asymptotic expansion for $f$ as $\lambda \to \infty$.

Now suppose that

$$f(\lambda,\alpha) = \int_0^1 e^{\lambda(-x+\alpha\sin x)}\,\mathrm{d}x ,$$

and that $\lambda \to \infty$. Obtain the leading order term for $f$

(i) if $0 \leqslant \alpha < 1$;

(ii) if $\alpha = 1$.

Deduce that there is a distinguished limit for which

$$\alpha = 1 - \nu/\lambda^p \qquad \text{and} \qquad f = h(\nu)/\lambda^q$$

where $p, q$ and the function $h(\nu)$ are to be determined. The result for $h$ may be left in integral form. Show that this result is consistent with (i) and (ii) in the appropriate limits.

$$\left[ Note : \Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t . \right]$$

**2**

Derive the leading order (WKB) solution of the problem for real $k$:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\, x^2} + k^2 y = 0$$

with $k = k(\epsilon x)$ and $\epsilon \to 0$. Write down the corresponding solution if $k^2 < 0$, and state, without detailed calculation, what happens if $k^2 = 0$ at some value of $\epsilon x$.

Hence find at leading order the eigenvalues $\lambda_n$ and eigenfunctions $y_n$ for the Sturm-Liouville problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\, x^2} + \lambda r(x) y = 0 \qquad \text{with} \quad y(0) = y(\pi) = 0$$

and $r(x) > 0$ in the limit $\lambda \to \infty$.

Compare the exact and approximate values of $\lambda_n$ for the case

$$(x+1)^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\, x^2} + \lambda y = 0 \qquad \text{with} \quad y(0) = y(\pi) = 0\,.$$

How large must $n$ be before the error in $\lambda_n$ is less than 1%?

**3**

(a) Describe the use of the Briggs-Bers technique to determine the long-time behaviour of solutions of linear initial-value problems, and in particular explain how to determine if a system is stable, convectively unstable or absolutely unstable. Illustrate your answer by considering the fourth-order equation

$$\frac{\partial A}{\partial t} + a_1 \frac{\partial^2 A}{\partial x^2} + a_2 \frac{\partial^4 A}{\partial x^4} + a_3 A = 0\,,$$

where $a_{1,2,3}$ are complex constants. Be careful to state the conditions under which Briggs-Bers can be applied in this case.

(b) Consider $y(x;\epsilon)$ satisfying

$$(\epsilon + x) \frac{\mathrm{d}y}{\mathrm{d}x} = \epsilon y \qquad \text{with } y(1) = 1\,.$$

Use the method of matched asymptotic expansions to calculate $y(0)$ correct up to and including $O(\epsilon)$. Obtain the exact solution, and verify that the exact value of $y(0)$ agrees with your approximate result to the appropriate order.

**4**

(a) Consider an inviscid fluid flowing in a straight channel parallel to the $x$ axis, with walls $y = y_1$ and $y = y_2$ with $y_1 < 0 < y_2$. The steady flow has velocity $U(y)$. The unsteady perturbation stream function is of the form $\psi(y)\exp(\mathrm{i}k(x - ct))$ with $k$ real, where $\psi(y)$ satisfies Rayleigh's equation

$$(U - c)\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}y^2} - k^2\psi\right) - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\,\psi = 0$$

with $\psi(y_1) = \psi(y_2) = 0$.

Prove:

(i) that if the flow is unstable then $U(y)$ must have an inflection point, i.e. $y = y_s$ with $y_1 \leqslant y_s \leqslant y_2$ such that $U''(y_s) = 0$;

(ii) that if the flow is unstable then

$$\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\,[U(y) - U(y_s)] < 0$$

for some $y_1 < y < y_2$.

(b) Consider the linear Ginzburg equation

$$\frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x} - \mu\eta - \gamma\frac{\partial^2\eta}{\partial x^2} = 0 \tag{1}$$

with $\mu = \mu_0 - \nu\epsilon^2x^2$ and other coefficients constant, where $\epsilon \ll 1$ and $\nu > 0$. By writing

$$\eta(x, t; \epsilon) = \exp(Ux/2\gamma)\,b(\xi)\exp(-\mathrm{i}\,\omega_0 t - \mathrm{i}\,\epsilon\omega_1 t)$$

with $\xi = \sqrt{\epsilon}fx$, show that

$$\omega_0 = \mathrm{i}\left(\mu_0 - \frac{U^2}{4\gamma}\right)$$

and

$$\frac{\mathrm{d}^2b}{\mathrm{d}\xi^2} + (\lambda - \xi^2)b = 0 \tag{2}$$

for suitable constants $f$ and $\lambda$ to be determined. Given that nontrivial solutions of equation (2) that tend to zero as $\xi \to \pm\infty$ exist only when $\lambda$ is a positive odd integer, show that equation (1) possesses a spatially bounded stable global mode if

$$\mu_0 \leqslant \frac{U^2}{4\gamma} + \epsilon\sqrt{\nu\gamma}\,.$$

**END OF PAPER**