UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 7 June, 2010 1:30 pm to 4:30 pm

PAPER 68

PERTURBATION AND STABILITY METHODS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Obtain two terms of an asymptotic expansion for each root of the equation

$$x^2 e^{-x} = \epsilon$$

for $\epsilon > 0$ and $\epsilon \to 0$.

(b) State Watson's lemma and sketch a proof of it.

Suppose that

$$f(\lambda) = \int_a^b e^{\lambda \phi(x)} g(x) \, \mathrm{d}x \, .$$

If ϕ is monotonic decreasing in $[a, b]$ and $\phi'(a) \neq 0$ and $g(a) \neq 0$, find two terms of an asymptotic expansion for f as $\lambda \to \infty$.

Now suppose that

$$f(\lambda, \alpha) = \int_0^1 e^{\lambda(-x + \alpha \sin x)} \, \mathrm{d}x \, ,$$

and that $\lambda \to \infty$. Obtain the leading order term for f

(i) if $0 \leqslant \alpha < 1$;

(ii) if $\alpha = 1$.

Deduce that there is a distinguished limit for which

$$\alpha = 1 - \nu/\lambda^p \qquad \text{and} \qquad f = h(\nu)/\lambda^q$$

where p, q and the function $h(\nu)$ are to be determined. The result for h may be left in integral form. Show that this result is consistent with (i) and (ii) in the appropriate limits.

$$\left[Note : \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t \, . \right]$$

2

Derive the leading order (WKB) solution of the problem for real k:

$$\frac{d^2y}{d\,x^2} + k^2 y = 0$$

with $k = k(\epsilon x)$ and $\epsilon \to 0$. Write down the corresponding solution if $k^2 < 0$, and state, without detailed calculation, what happens if $k^2 = 0$ at some value of ϵx.

Hence find at leading order the eigenvalues λ_n and eigenfunctions y_n for the Sturm-Liouville problem

$$\frac{d^2y}{d\,x^2} + \lambda r(x) y = 0 \qquad \text{with} \quad y(0) = y(\pi) = 0$$

and $r(x) > 0$ in the limit $\lambda \to \infty$.

Compare the exact and approximate values of λ_n for the case

$$(x+1)^2 \frac{d^2y}{d\,x^2} + \lambda y = 0 \qquad \text{with} \quad y(0) = y(\pi) = 0\,.$$

How large must n be before the error in λ_n is less than 1% ?

3

(a) Describe the use of the Briggs-Bers technique to determine the long-time behaviour of solutions of linear initial-value problems, and in particular explain how to determine if a system is stable, convectively unstable or absolutely unstable. Illustrate your answer by considering the fourth-order equation

$$\frac{\partial A}{\partial t} + a_1 \frac{\partial^2 A}{\partial x^2} + a_2 \frac{\partial^4 A}{\partial x^4} + a_3 A = 0\,,$$

where $a_{1,2,3}$ are complex constants. Be careful to state the conditions under which Briggs-Bers can be applied in this case.

(b) Consider $y(x;\epsilon)$ satisfying

$$(\epsilon + x)\frac{dy}{dx} = \epsilon y \qquad \text{with } y(1) = 1\,.$$

Use the method of matched asymptotic expansions to calculate $y(0)$ correct up to and including $O(\epsilon)$. Obtain the exact solution, and verify that the exact value of $y(0)$ agrees with your approximate result to the appropriate order.

4

(a) Consider an inviscid fluid flowing in a straight channel parallel to the x axis, with walls $y = y_1$ and $y = y_2$ with $y_1 < 0 < y_2$. The steady flow has velocity $U(y)$. The unsteady perturbation stream function is of the form $\psi(y) \exp(\mathrm{i}k(x - ct))$ with k real, where $\psi(y)$ satisfies Rayleigh's equation

$$(U - c)\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}y^2} - k^2\psi\right) - \frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\,\psi = 0$$

with $\psi(y_1) = \psi(y_2) = 0$.

Prove:

(i) that if the flow is unstable then $U(y)$ must have an inflection point, i.e. $y = y_s$ with $y_1 \leqslant y_s \leqslant y_2$ such that $U''(y_s) = 0$;

(ii) that if the flow is unstable then

$$\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\,[U(y) - U(y_s)] < 0$$

for some $y_1 < y < y_2$.

(b) Consider the linear Ginzburg equation

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} - \mu\eta - \gamma\frac{\partial^2 \eta}{\partial x^2} = 0 \qquad (1)$$

with $\mu = \mu_0 - \nu\epsilon^2 x^2$ and other coefficients constant, where $\epsilon \ll 1$ and $\nu > 0$. By writing

$$\eta(x, t; \epsilon) = \exp(Ux/2\gamma)\, b(\xi) \exp(-\mathrm{i}\,\omega_0 t - \mathrm{i}\,\epsilon\omega_1 t)$$

with $\xi = \sqrt{\epsilon} f x$, show that

$$\omega_0 = \mathrm{i}\left(\mu_0 - \frac{U^2}{4\gamma}\right)$$

and

$$\frac{\mathrm{d}^2 b}{\mathrm{d}\xi^2} + (\lambda - \xi^2) b = 0 \qquad (2)$$

for suitable constants f and λ to be determined. Given that nontrivial solutions of equation (2) that tend to zero as $\xi \to \pm\infty$ exist only when λ is a positive odd integer, show that equation (1) possesses a spatially bounded stable global mode if

$$\mu_0 \leqslant \frac{U^2}{4\gamma} + \epsilon\sqrt{\nu\gamma}\,.$$

END OF PAPER