UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2010 1:30 pm to 4:30 pm

PAPER 67

BIOLOGICAL PHYSICS

Attempt all ***THREE*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a rod of length L, Young's modulus E, mass per unit length λ, and an elliptical cross-section whose principal components of the moment of inertia tensor are $I_2 < I_1$ so it is easier to bend in one direction than the other. If it is held vertically, clamped at the bottom end ($z = 0$), show that the equation governing small amplitude deflections $X(z)$ (in the easy direction) from the vertical under the action of gravity is

$$I_2 E X_{zzzz} - (T X_z)_z = 0,$$

where $T(z)$ is the internal tension, which you should find. If the upper end is free, write down the complete set of boundary conditions on $X(z)$. Show that the function $u(z) = X_z(z)$ admits a similarity solution of the form

$$u = \eta^{1/3} F(\eta)$$

where

$$\eta = \frac{2}{3}\left[\lambda g \,(L - z)^3 / E I_2\right]^{1/2}.$$

Noting that the differential equation for Bessel functions J_ν has the form

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0,$$

show that $F = a\, J_{-1/3} + b\, J_{1/3}$, and find the associated boundary conditions on the function u, using the asymptotic form $J_\nu(x) \sim x^\nu$ in the limit $x \to 0$. Find the critical condition for the rod to buckle under its own weight.

2

Consider an elastic filament in two dimensions that has an intrinsic curvature $\kappa_0(s)$, overall length $\mathcal{L}$ and projected length L along the x-axis, from which it deviates by the small amplitude function $\zeta(x)$. If it subjected to a force F of extension along x, find the appropriate quadratic energy functional for the filament. From the Euler-Lagrange equation, relate the Fourier transforms of ζ and κ_0 and thereby deduce the relationship between F and the length deficit $\mathcal{L} - L$. Specialize to the case of large F and express your result in a form appropriate to measurements on an ensemble of filaments, each having a realization of the function $\kappa_0(s)$.

Contrast the force-extension behaviour of the above randomly curved polymer with that of a freely jointed chain composed of N segments of length b, subject to an extensional force F. You should derive the exact force-displacement relationship and then deduce the limiting form at high extension.

3

An activator-inhibitor reaction diffusion system in dimensionless form is given by

$$u_t = u_{xx} + \frac{u^2}{v} - bu$$

$$v_t = dv_{xx} + u^2 - v$$

where b and d are positive constants. Determine the positive steady states and show, by an examination of the eigenvalues in a linear stability analysis of the diffusionless situation, that the reaction kinetics cannot exhibit oscillatory solutions if $b < 1$.

Determine the conditions for the steady state to be driven unstable by diffusion. Show that the parameter domain for diffusion-driven instability is given by $0 < b < 1$, $db > 3 + 2\sqrt{2}$ and sketch the (b, d) parameter space in which diffusion-driven instability occurs. Further, show that at the bifurcation to such an instability the critical wave number k_c is given by $k_c^2 = (1 + \sqrt{2})/d$.

END OF PAPER