UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 1 June, 2010 9:00 am to 12:00 pm

PAPER 66

SLOW VISCOUS FLOW

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Use the Papkovich–Neuber representation of Stokes flow to find the velocity field $\mathbf{u}$ due to a rigid sphere of radius a moving with velocity $\mathbf{V}$ and no rotation through unbounded viscous fluid.

Show that

$$\frac{\partial \mathbf{u}}{\partial r} = \frac{3\mathbf{V}}{2a}\cdot\left(\mathbf{nn} - \mathbf{I}\right) \qquad \text{on} \quad r = a\,,$$

where $\mathbf{n}$ is the normal to the sphere and $\mathbf{I}$ is the identity tensor.

On $r = a$ the stress tensor is given by

$$\boldsymbol{\sigma} = \frac{3\mu}{2a}\left[(\mathbf{V}\cdot\mathbf{n})(2\mathbf{nn} - \mathbf{I}) - \mathbf{Vn} - \mathbf{nV}\right].$$

Calculate the force on the sphere.

(b) Now consider a nearly spherical rigid particle with radius $r = a + \epsilon f(\theta, \phi)$, where $\epsilon \ll 1$, moving under the influence of a fixed force $\mathbf{F}$ and zero couple through unbounded viscous fluid. The velocity of the particle $\mathbf{U} + \boldsymbol{\Omega} \wedge \mathbf{x}$ can be expanded as a regular perturbation series

$$\mathbf{U} = \mathbf{U}_0 + \epsilon\,\mathbf{U}_1 + O(\epsilon^2) \qquad \text{and} \qquad \boldsymbol{\Omega} = \boldsymbol{\Omega}_0 + \epsilon\,\boldsymbol{\Omega}_1 + O(\epsilon^2)$$

with similar expansions for the velocity field $\mathbf{u}$ and stress field $\boldsymbol{\sigma}$. What are the values of $\mathbf{U}_0$ and $\boldsymbol{\Omega}_0$?

What are the exact boundary conditions satisfied by $\mathbf{u}$ and $\boldsymbol{\sigma}$ on the surface S of the particle? By expanding these conditions in powers of ϵ, deduce that

$$\mathbf{u}_1 = \mathbf{U}_1 + \boldsymbol{\Omega}_1 \wedge \mathbf{x} - f\,\frac{\partial \mathbf{u}_0}{\partial r} \qquad \text{on} \quad r = a$$

and

$$\int_{r=a} \boldsymbol{\sigma}_1\cdot\mathbf{n}\,dS = \int_{r=a} \mathbf{x} \wedge \boldsymbol{\sigma}_1\cdot\mathbf{n}\,dS = \mathbf{0}\,.$$

Explain why there are no terms involving f and $\boldsymbol{\sigma}_0$ in the two integrals.

(c) Use the reciprocal theorem on $\mathbf{u}_1$ and a suitable test flow $\hat{\mathbf{u}}$ to obtain an equation for $\mathbf{U}_1$. For the case of a slightly ellipsoidal particle with $f = a(\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n})$, where $\mathbf{D}$ is a constant, symmetric, traceless, second-rank tensor, show that

$$\mathbf{U}_1 = \frac{1}{5}\,\mathbf{U}_0\cdot\mathbf{D}\,.$$

Explain briefly why $\boldsymbol{\Omega}_1 = \mathbf{0}$ for such a particle.

$\Big[$*You may assume that* $\displaystyle\int_{r=a} n_i\, n_j\, n_k\, n_l\, dS = \frac{4\pi a^2}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right).\Big]$

2 A long straight horizontal channel aligned with the x-axis has a V-shaped cross-section defined by a rigid boundary at $z = |y| \cot\alpha$, where y is the cross-channel coordinate and z is the vertical coordinate. Fluid with kinematic viscosity ν occupies the bottom of the channel with depth $h(x,y,t)$ so that the height ζ of the free surface is given by $\zeta = h(x,y,t) + |y|\cot\alpha$. Surface tension is negligible and the flow is driven by gravity.

Assume that the midline depth $h_0 = h(x,0,t)$ can be taken as a representative lengthscale for both the depth and the width of the flow. Use scaling arguments to show that (i) variations in free-surface height ζ across the channel flatten on a timescale $\tau_y = \nu/gh_0$ and (ii) variations in free-surface height along the channel over a lengthscale L, where $L \gg h_0$, flatten on a timescale $\tau_x = \nu L^2/gh_0^3$. [*Hint: estimate the horizontal and vertical velocities from the pressure gradient.*] Why is it reasonable to assume that ζ is approximately independent of y when calculating the evolution of flow along the channel?

The midline depth satisfies $|\partial h_0/\partial x| \ll 1$. Explaining any approximations made, show that

$$\tan\alpha \, \frac{\partial h_0^2}{\partial t} = \frac{g\,Q(\alpha)}{\nu} \frac{\partial}{\partial x}\left(h_0^4 \frac{\partial h_0}{\partial x}\right),$$

where $Q(\alpha)$ is the area integral of the solution of $\nabla^2\phi = -1$ in a two-dimensional domain and with boundary conditions that should each be specified. [*You should not make any assumptions about the value of α, but explicit solution for ϕ is not required.*]

A fixed volume V of fluid is released at the origin at $t = 0$ and spreads in both directions along the channel. Calculate the corresponding large-time similarity solution and show that the nose of the current is at

$$x_N(t) = \left(\frac{V}{I\tan\alpha}\right)^{3/7}\left(\frac{7g\,Q(\alpha)t}{3\,\nu\tan\alpha}\right)^{2/7}, \qquad \text{where} \quad I = \int_{-1}^{1}(1-u^2)^{2/3}\,du\,.$$

Sketch the qualitative variation of $h_0(x^*,t)$ at a fixed position x^* for $t > 0$. Using similarity variables or otherwise, show that the maximum depth at x^* is attained when $x_N^2 = \frac{7}{3}x^{*2}$.

When the level of fluid is falling, fluid must drain down the walls of the channel. Sketch a cross-section through the channel showing the expected distribution of fluid when the midline depth has decreased to half its maximum value for, say, $\alpha \approx \frac{\pi}{4}$. Comment on whether this distribution invalidates the above calculation of $x_N(t)$ for large t.

3 A planar sheet of fluid of viscosity μ undergoes extension. With respect to Cartesian axes, the sheet occupies $-\frac{1}{2}\,h(x,t) \leqslant z \leqslant \frac{1}{2}\,h(x,t)$. There is no flow or variation in the y-direction, so that the velocity $\mathbf{u}(x,z,t) = (u,0,w)$. The sheet is acted upon by surface tension, with constant coefficient γ, but the effect of gravity is negligible.

Assuming that $\partial h/\partial x \ll 1$, explain why u is approximately independent of z and derive equations for $w(x,z,t)$ and for the z-component of the stress tensor σ_{zz}. Deduce that

$$\sigma_{xx} = -p_a + \frac{1}{2}\gamma\frac{\partial^2 h}{\partial x^2} + 4\,\mu\,\frac{\partial u}{\partial x}\,,$$

where p_a is the uniform pressure outside the sheet.

Draw a diagram to show all of the forces acting on a fluid slice of length δx and varying thickness. Deduce that

$$\frac{\partial}{\partial x}\left(4\,\mu h\,\frac{\partial u}{\partial x}\right) + \frac{1}{2}\gamma h\,\frac{\partial^3 h}{\partial x^3} = 0 \qquad (1)$$

and obtain another relationship between $h(x,t)$ and $u(x,t)$.

Two cylindrical gas bubbles of radius a are gently pressed against one another by a weak external flow in the surrounding liquid. Deformation of the bubbles is negligible except in a flat region of fixed length $2L$ where the bubbles are separated by a thin liquid sheet of thickness $h_0(x,t) \ll L$. Making reference to the pressure in the flat region and in the external fluid, explain why the liquid drains out of the sheet.

Within the flat region, the sheet thickness h_0 is initially independent of x. Using (1), show that $u = Ux/L$ for some $U(t)$. Deduce that h_0 remains independent of x and deduce an ordinary differential equation for $h_0(t)$.

The value of U is controlled by short transition regions at each end of the sheet over which the interfacial curvature changes from 0 to $1/a$. The lengthscale δ of these regions satisfies $h_0 \ll \delta \ll L$. Use scaling arguments to show that:

(i) $\delta \sim (ah_0)^{1/2}$, (ii) $U \sim (\gamma/\mu)(h_0/a)^{1/2}$, and

(iii) uh is approximately constant throughout the transition region.

Use a scaled co-ordinate $\xi = (x-L)/(ah_0)^{1/2}$ to rescale (1) in the transition region. Eliminate u to obtain a third-order differential equation for $h(\xi)$. Integrate this twice to show that

$$h^{-1/2}\,\frac{dh}{d\xi} = \frac{16\mu U}{3\gamma}\,a^{1/2}\left[1-(h_0/h)^{3/2}\right].$$

By considering the behaviour of h for $h \gg h_0$, show that

$$U = \frac{3\gamma}{8\mu}\left(\frac{h_0}{a}\right)^{1/2}.$$

Hence find $h_0(t)$. Comment on the large-time behaviour.

4 A rigid sphere, radius a, moves through a fluid of viscosity μ towards a rigid, horizontal plane. The minimum distance between the sphere and the plane is $h_0(t) \ll a$. Assuming lubrication theory, but explaining any geometrical approximations, show that the pressure in the fluid is

$$p(r,t) \;=\; -3\mu a \dot{h}_0 \left(h_0 + \frac{r^2}{2a} \right)^{-2} ,$$

where a dot denotes differentiation with respect to time and r is the horizontal distance from the vertical line through the centre of the sphere.

The horizontal plane is now coated with a thin elastic layer whose surface deformation is given by $\delta h = -\eta p(r,t)$, where the compliance η is a constant. In what follows, use non-dimensional variables

$$H_0 \equiv \frac{h_0}{\hat{h}}, \quad \dot{H}_0 \equiv \frac{\dot{h}_0}{\hat{v}}, \quad R \equiv \frac{r}{\sqrt{a\hat{h}}}, \quad T \equiv \frac{t\hat{v}}{\hat{h}}, \quad P \equiv \frac{p\hat{h}^2}{\mu a \hat{v}},$$

where $\hat{h}$ and $\hat{v}$ are typical scales for $h_0(t)$ and $\dot{h}_0(t)$. Why are the scales used to define R, T and P appropriate? What is the non-dimensional compliance ϵ of the elastic layer?

Assuming that $\epsilon \ll 1$, expand P as a power series $P = P_0 + \epsilon P_1 + \dots$. State the solution for $P_0(R,T)$ and show that

$$P_1(R,T) \;=\; \frac{k(T)}{H_0^3} I_3(\xi) + 18 \frac{\ddot{H}_0}{H_0^4} I_4(\xi) - 18 \frac{\dot{H}_0^2}{H_0^5} I_5(\xi) - \frac{54}{5} \frac{\dot{H}_0^2}{H_0^5} (1+\xi)^{-5} ,$$

where $k(T)$ is some function of T, $\xi = R^2/2H_0$ and

$$I_n(x) \;\equiv\; -\int_x^\infty \frac{dy}{y(1+y)^n} \;=\; \log\left(\frac{x}{1+x} \right) + \sum_{i=1}^{n-1} \frac{1}{i} \frac{1}{(1+x)^i} .$$

Explain why regularity of the pressure requires that

$$k(T) \;=\; 18 \left(\frac{\dot{H}_0^2}{H_0^2} - \frac{\ddot{H}_0}{H_0} \right) .$$

Simplify the equation for $P_1(R,T)$ using the expansions of $I_n(x)$.

Show that the non-dimensional hydrodynamic lift force is given by

$$F \;=\; 6\pi \left[-\frac{\dot{H}_0}{H_0} + \epsilon \left(\frac{\ddot{H}_0}{H_0^3} - \frac{12}{5} \frac{\dot{H}_0^2}{H_0^4} \right) + \dots \right] .$$

The sphere is sedimenting due to its weight. Does the presence of an elastic layer initially increase or decrease the rate of sedimentation? What is the physical reason for this?

Determine the order-of-magnitude gap thickness H_0^* at which the $O(\epsilon)$ term in the series expansion for F becomes comparable to the leading-order term. What does this breakdown of the expansion correspond to physically?

END OF PAPER