UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 2 June, 2010 1:30 pm to 4:30 pm

PAPER 65

FLUID DYNAMICS OF ENERGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

Candidates may bring handwritten or personally typed lecture notes and handouts only into the examination. No other photocopies of published material are allowed.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A dense fluid of density ρ and viscosity μ migrates along the lower boundary of a porous medium of permeability k, porosity ϕ originally filled with fluid of density ρ_2. The fluid migrates as a two dimensional current from a uniform line source. If the layer of original fluid in the porous medium is deep show that the current dynamics are described by the equation

$$\frac{\partial h}{\partial t} = \lambda \frac{\partial}{\partial x}\left[h\frac{\partial h}{\partial x}\right]$$

where $\lambda = k(\rho - \rho_2)g/\phi\mu\,,$ x is the distance along the boundary in the direction of the current and h is the depth of the fluid. You may use the approximation of a long thin current.

Derive a solution in the case of a finite release of fluid of volume V per unit distance in the cross-flow direction.

If the fluid slowly drains through the boundary of the domain at a rate $\phi\Gamma h$ per unit area, show that the governing equation has the new form

$$\frac{\partial h}{\partial t} = \lambda \frac{\partial}{\partial x}\left[h\,\frac{\partial h}{\partial x}\right] - \Gamma h\,.$$

Using a suitable transform of time, derive a solution for the motion of an initial volume of fluid V as it spreads and drains.

Interpret this solution.

2

A room is heated from below with uniform heat load Q and has a ventilation opening in the floor and the ceiling each of area A, and a vertical distance H apart. Show that in the absence of wind effects, the ventilation flow rate is given by (approximately)

$$V = cA^{2/3}\left[\frac{gHQ}{2\,\rho\, C_p T_e}\right]^{1/3},$$

where c is a loss coefficient for the openings and T_e is the external temperature, with g the acceleration of gravity and ρ and C_p are the density and the specific heat capacity of the air within the room.

If the room is ventilating in steady state, and the heating load is then reduced to the value λQ, show that a transient two layer stratification becomes established, in which the depth h of the lower layer satisfies an equation of the form

$$\rho\, C_p A_r \frac{d}{dt}\left[h(\Delta T - \Delta T_0)\right] = -\rho\, C_p V \Delta T_0 + \lambda Q\,, \qquad (*)$$

where $\Delta T = (T - T_e)$ is the temperature contrast with the exterior and $\Delta T_0 = (T_0 - T_e)$ is the initial temperature contrast with the exterior, and in this expression V is the ventilation rate in the room for which you should give an expression, and A_r is the cross-sectional area of the room.

Show that with suitable scaling, the equation $(*)$ can be written in the dimensionless form

$$\frac{dx}{ds} = \sqrt{1-x} - \lambda\,,$$

where

$$x = \frac{h}{H}\left[1 - \frac{\Delta T}{\Delta T_0}\right],$$

and s is the scaled time.

Using the expression for the ventilation rate V, show that the new lower layer eventually fills the depth of the room, $h = H$. Derive the equation for the temperature evolution of the room once the front has left the room and find the final equilibrium temperature of the room.

3

Consider a statistically steady, two-dimensional incompressible flow, where x is the streamwise direction, and y is the vertical direction. Decompose the velocity $\mathbf{u} = (u, v)$ into a time-averaged mean $\overline{\mathbf{u}} = (U(x,y), V(x,y))$ and a perturbation $\hat{\mathbf{u}} = (\hat{u}(x,y,t), \hat{v}(x,y,t))$.

(a) Show that

$$\begin{aligned}\frac{\partial}{\partial x}U + \frac{\partial}{\partial y}V &= 0\,,\\ U\frac{\partial}{\partial x}U + V\frac{\partial}{\partial y}U &= -\frac{1}{\rho}\frac{\partial}{\partial x}\bar{p} + \nu\frac{\partial^2}{\partial y^2}U - \frac{\partial}{\partial y}\overline{\hat{u}\hat{v}} - \frac{\partial}{\partial x}\overline{\hat{u}^2} + \nu\frac{\partial^2}{\partial x^2}U\,,\end{aligned}$$

where ρ is the density, $\bar{p}$ is the time-averaged pressure, and ν is the kinematic viscosity.

(b) As an example, consider the flow associated with a jet issuing from a thin slot, centred at $(0,0)$, into quiescent fluid. If the flow is at sufficiently high speed, present scaling arguments to justify the reduction of the streamwise momentum equation to

$$U\frac{\partial}{\partial x}U + V\frac{\partial}{\partial y}U = -\frac{\partial}{\partial y}\overline{\hat{u}\hat{v}} = -\mathrm{sgn}(y)\frac{\partial}{\partial y}\left[l^2(\partial U/\partial y)^2\right]\,, \qquad (1)$$

where l is a "mixing length" which you should define carefully.

(c) Present a physical argument why $l = C_1|y|$ is an appropriate assumption for the mixing length, for C_1 a constant.

(d) Show that equation (1) implies constant (streamwise) specific momentum flux per unit width M_0, i.e.

$$\frac{d}{dx}M_0 = \frac{d}{dx}\left(\int_{-\infty}^{\infty} U^2\,dy\right) = 0\,.$$

(e) By considering a streamfunction ψ, show that there exists a similarity solution for the turbulent plane jet with

$$\psi = M_0^{1/2}\,x^{1/2}f(y/x) = M_0^{1/2}\,x^{1/2}f(\eta)\,.$$

Define the edges of the jet as the (constant) values of $\eta = \pm\eta_w = \pm y_w/x$ where $|\partial U/\partial y|$ is maximum, and $U = 0$. Hence, derive the two equations which must be satisfied by f to be consistent with equations (1) and (2).

(f) Show that the eddy viscosity $\nu_T(x,y)$ in the jet increases like $x^{1/2}$ with downstream distance, and is maximum at the edge of the jet $y = \pm y_w$ (under the assumptions above). Hence show that an "effective" Reynolds number $U(x,0)y_w/\nu_T(x,\pm y_w)$ does not depend on downstream distance.

4

Consider a cylindrical tank, of depth H and radius R. Define the upper surface as $z = 0$, with z increasing downwards. Initially there is a layer of fluid of density $\rho_u(0)$ between $0 < z < h(0)$, and a layer of fluid of density $\rho_l > \rho_u(0)$ between $h < z < H$. The lid of the tank is driven at a steady rotation rate Ω. The induced flow is such that the upper layer deepens with time-dependent depth $h(t)$, but remains well-mixed, with uniform density $\rho_u(t) \geqslant \rho_u(0)$, while the lower layer remains stationary. The small interfacial depth d_I over which the density varies substantially between the two layers is close to constant with time. For some drag coefficient c_D, the interfacial stress is $\tau = c_D \rho_l u_I^2$, where u_I is a characteristic (radial) velocity in the upper layer in the vicinity of the interface. Always assume that the rate of increase of potential energy is proportional to the rate of work done on the upper layer by the lid, with constant of proportionality C_W, and that the Boussinesq approximation is valid.

(a) Show that $g(\rho_l - \rho_u)h/\rho_l \equiv g'h$ remains a constant, and hence that the rate of increase of the potential energy of the system is directly proportional to the rate of increase of the depth of the upper layer.

(b) Consider model "S", which is defined by the assumption that the upper layer is in time-independent and depth-independent solid body rotation (i.e. assume that $u_I = \Omega R$). Show that the upper layer deepens at a constant rate proportional to the inverse of the bulk Richardson number $Ri_B = g'h/(\Omega^2 R^2)$.

(c) Now consider model "K", which is defined by the assumption that the total kinetic energy of the upper layer remains constant as it deepens (while still remaining in solid body rotation). Show that this implies that $u_U^2 h = \Omega^2 R^2 h(0) C_K^2$ for some constant C_K, where u_U is the rms velocity of the upper layer. With the further assumption that $u_I = C_I u_U$ for some constant C_I, show that

$$\frac{h}{h(0)} = (1 + A\Omega\, t)^{2/5} \ ,$$

for some constant A, which you should determine.

(d) Define the flux Richardson number. By considering the interfacial Richardson number $Ri_I = g' d_I / u_I^2$, discuss briefly the appropriateness of the fundamental assumption of constant C_W for model "S" and model "K". Furthermore, by considering the energetics of entrainment, show that you expect the actual evolution of the deepening of the upper layer to be somewhere between the predictions of model "S" and model "K".

END OF PAPER