UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 27 May, 2010 9:00 am to 11:00 am

PAPER 64

REACTION-DIFFUSION EQUATIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

On a Hilbert space $\mathcal{H}$ consider the abstract linear problem

$$\begin{aligned} \frac{d}{dt}u(t) &= Lu(t) + f(t)\,, \qquad\qquad t \in (0,T]\,, \quad f:[0,T] \mapsto \mathcal{H}(\Omega)\,, \\ u(0) &= u_0\,. \end{aligned}$$

State assumptions on the operator $L : D(L) \subset \mathcal{H} \mapsto \mathcal{H}$ and on f in order to formulate and prove local existence of a classical solution (Please define).

Sketch the existence proof in a way which makes it clear why the assumption on f is needed.

2

On a bounded, smooth domain $\Omega \subset \mathbb{R}^n, n \in \mathbb{N}$ consider the problem

$$\begin{aligned} \partial_t u &= \Delta u + \lambda u - u^3\,, && x \in \Omega\,,\ t > 0\,, \\ u &= 0\,, && x \in \partial\Omega\,,\ t > 0\,, \\ m &\leqslant u(x,0) \leqslant M\,, \quad u(x,0) \in \mathcal{H}_\alpha\,,\ \alpha > 0 && x \in \Omega\,. \end{aligned}$$

Suppose the existence of a local classical solution $u(t)$, which satisfies moreover $\frac{du}{dt} \in C\big((0,t_0], \mathcal{H}_\delta\big)$ with $\frac{n}{4} < \delta$ and $u(t) \in C\big([0,t_0]\,, \mathcal{H}_\alpha\big)$.

Show that this solution satisfies indeed (at least) $u(t) \in C^1((0,t_0], C^2(\bar{\Omega}))$. Moreover, show that this solution is uniformly bounded and thus can be extended globally.

3

For a constant $k > 0$ consider the equation

$$\partial_t u = \partial_{xx} u + u(k-u)\,, \qquad x \in \mathbb{R}\,,\ t > 0\,.$$

Show the existence of non-negative travelling waves $u(x,t) = w(x-ct)$ for $c \geqslant 0$ via phase plane analysis.

4

Consider a travelling wave solution $w_c(z)$, $z = x - ct$ of the equation

$$\partial_t u = \partial_{xx} u + u(k - u)\,, \qquad x \in \mathbb{R},\ t > 0\,,$$

with $k > 0$ and for $c \geqslant c_*$ where $c_* > 0$ is given.

What sort of stability of travelling waves can be expected? Which equation is (usually) the first one to consider? Use a theorem of the lecture in order to show stability of the essential spectrum under perturbations in the weighted spaces

$$L^2_\gamma \;=\; \{u \in L^2(\mathbb{R}) : \|u\|^2_{2,\gamma} \;=\; \int_{\mathbb{R}} |u|^2\, e^{\,2\,\gamma x}\, dx \;<\; \infty\}\,,$$

for $\gamma > 0$ where γ is to be determined.

END OF PAPER