UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2010 1:30 pm to 4:30 pm

PAPER 63

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

*Attempt **THREE** questions from Section I*
*and **ONE** question from Section II.*

*There are **SEVEN** questions in total.*

The questions in Section II carry twice the weight of those in Section I.
Questions in each Section carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1

Given the ODE system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, where f is smoothly differentiable, we consider the two-step method

$$\mathbf{y}_{n+2} - \tfrac{6}{7}\,h\mathbf{f}(\mathbf{y}_{n+2}) + \tfrac{2}{7}\,h^2\,\frac{\partial \mathbf{f}(\mathbf{y}_{n+2})}{\partial \mathbf{y}}\,\mathbf{f}(\mathbf{y}_{n+2}) = \tfrac{8}{7}\,\mathbf{y}_{n+1} - \tfrac{1}{7}\,\mathbf{y}_n.$$

1. Determine the order of this method.

2. Is it A-stable?

2

1. Given a Hilbert space $\mathbb{H}$ and a linear operator $\mathcal{L} : \mathbb{H} \to \mathbb{H}$, what is meant by $\mathcal{L}$ being positive definite?

2. Prove that if $\mathcal{L}$ is positive definite then $\mathcal{L}u = f$ is the Euler–Lagrange equation of the variational problem $I(v) = \langle \mathcal{L}v, v\rangle - 2\langle f, v\rangle$.

3. Suppose that $\mathbb{H}$ consists of the closure of twice smoothly-differentiable functions that satisfy the zero boundary conditions $u(\pm 1) = u'(\pm 1) = 0$ and is equipped with the standard L_2 inner product. In addition, $p, q \in \mathbb{H}$ are given, such that $p(x) > 0$, $q(x) \geqslant 0$, $x \in (-1, 1)$. Prove that the operator

$$\mathcal{L}[f] = (pu'')'' + qu$$

is positive definite.

3

Consider the Cauchy problem for the two-dimensional advection equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}, \qquad u = u(x, y, t), \quad (x, y) \in \mathbb{R}^2, \quad t \geqslant 0$$

and the semi-discretized scheme

$$u'_{m,k} = \frac{1}{\Delta x}\left(-\tfrac{3}{2}\, u_{m,k} - \tfrac{1}{2}\, u_{m-1,k} + \tfrac{1}{2}\, u_{m+1,k} + 2u_{m,k+1} - \tfrac{1}{2}u_{m,k+2}\right), \qquad m,\, k \in \mathbb{Z},$$

where $u_{m,k} \approx u(m\Delta x,\, k\Delta x,\, t)$.

1. What is the order of the method?
2. Is the method stable?

4

Let distinct $c_1, c_2, \ldots, c_s \in [0, 1]$ be given.

1. Define a collocation method with the nodes $c_1, \ldots, c_s$.
2. Prove that such a method can be reformulated as a Runge–Kutta method.
3. State a theorem on the order of a collocation method.
4. Prove that the Runge–Kutta method with the Butcher tableau

$$\begin{array}{r|ll} 1/3 & 5/12 & -1/12 \\ 1 & 3/4 & 3/4 \\ \hline & 3/4 & 1/4 \end{array}$$

is equivalent to collocation and determine its order.

5

Let $a \in \mathrm{C}^1([0,1]^2)$ and assume that $a(x,y) > 0$ for all $(x,y) \in [0,1]^2$. We consider the diffusion equation with variable diffusion coefficient,

$$\frac{\partial u}{\partial t} = \nabla^\top(a\nabla u), \qquad (x,y) \in [0,1]^2, \quad t \geqslant 0,$$

with initial conditions at $t = 0$ and zero Dirichlet boundary conditions.

1. The equation is computed by means of the semi-discretized set of ODEs

$$\begin{aligned} u'_{m,k} = \frac{1}{(\Delta x)^2} \Big[& a_{m-\frac{1}{2},k}\, u_{m-1,k} + a_{m+\frac{1}{2},k} u_{m+1,k} + a_{m,k-\frac{1}{2}}\, u_{m,k-1} \\ & + a_{m,k+\frac{1}{2}}\, u_{m,k+1} \\ & - (a_{m-\frac{1}{2},k} + a_{m+\frac{1}{2},k} + a_{m,k-\frac{1}{2}} + a_{m,k+\frac{1}{2}}) u_{m,k} \Big], \end{aligned}$$

where $u_{m,k} \approx u(m\Delta x,\, k\Delta x,\, t)$ and $k, l = 1, \ldots, N-1$, where $N\Delta x = 1$.

Prove that the method is stable.

2. We rewrite the above semidiscretized ODEs in the form

$$\begin{aligned} u'_{m,k} = {} & \frac{1}{(\Delta x)^2} \Big[a_{m-\frac{1}{2},k}\, u_{m-1,k} - (a_{m-\frac{1}{2},k} + a_{m+\frac{1}{2},k})\, u_{m,k} \\ & \qquad\qquad + a_{m+\frac{1}{2},k}\, u_{m+1,k} \Big] \\ & + \frac{1}{(\Delta x)^2} \Big[a_{m,k-\frac{1}{2}}\, u_{m,k-1} - (a_{m,k-\frac{1}{2}} + a_{m,k+\frac{1}{2}})\, u_{m,k} \\ & \qquad\qquad + a_{m,k+\frac{1}{2}}\, u_{m,k+1} \Big]. \end{aligned}$$

Written in a matrix form, this reads

$$\mathbf{u}' = (A + B)\mathbf{u},$$

where A and B are appropriate $N^2 \times N^2$ matrices. Prove that the splitting

$$\mathbf{u}^{n+1} = e^{(\Delta t)A}\, e^{(\Delta t)B}\mathbf{u}^n,$$

where $u^n_{m,k} \approx u_{m,k}(n\Delta t)$, approximates the semidiscretized ODEs with an error of $O(\Delta t)$.

3. We approximate each exponential by the [1/1] Padé approximation (in other words, use the split form of Crank–Nicolson). Prove that the ensuing full discretization is stable for all Courant numbers $\mu = \Delta t/(\Delta x)^2$.

SECTION II

6

Write an essay on A-stability of methods for ordinary differential equations.

7

Write an essay on stability analysis of numerical methods for partial differential equations of evolution using eigenvalue analysis.

END OF PAPER