UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 1 June, 2010 9:00 am to 12:00 pm

PAPER 62

APPROXIMATION THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) For $f \in C[0,1]$, write down the definition of the Bernstein polynomial $B_n(f)$ of degree n, and prove that $\|B_n(f)\|_\infty \leqslant \|f\|_\infty$.

(b) Let $f_{n0} \equiv 1$ and

$$f_{nm}(x) := x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)\cdots\left(x - \frac{m-1}{n}\right), \qquad 1 \leqslant m \leqslant n.$$

Show that $B_n(f_{nm}, x) = f_{nm}(1)x^m$.

(c) Using (a)-(b) prove that $B_n(e_m) \to e_m$ uniformly for any monomial $e_m(x) = x^m$.

(d) Quoting any appropriate theorem, derive that $B_n(f) \to f$ as $n \to \infty$ for any continuous $f \in C[0,1]$.

2

(a) Let σ_n be the Fejer operator, i.e., for a 2π-periodic function $f \in C(\mathbb{T})$,

$$\sigma_n(f,x) = \int_{-\pi}^{\pi} f(x-t)\,F_n(t)\,dt, \qquad F_n(t) := \frac{1}{\pi}\frac{1}{2n}\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}, \qquad \int_{-\pi}^{\pi} F_n(t)\,dt = 1.$$

Prove the estimate

$$\|\sigma_n(f) - f\|_\infty \leqslant c\,\omega_2(f, \tfrac{1}{\sqrt{n}}),$$

where $\omega_2(f,\delta)$ is the second modulus of smoothness of f. Hence prove that if f'' is continuous, then

$$\|\sigma_n(f) - f\| = \mathcal{O}(\tfrac{1}{n}).$$

(You should quote the relevant properties of the modulus $\omega_2(f,t)$ when using.)

(b) By considering $f(x) = \cos kx$ show that we cannot have a small-o estimate

$$\|\sigma_n(f) - f\| = o(\tfrac{1}{n})$$

for all $f \in C^2(\mathbb{T})$.

3

Let T_n be the Chebyshev polynomial of degree n:

$$T_n(x) = \cos n \arccos x\,, \qquad x \in [-1,1]\,.$$

(a) Prove that

$$1 - T_n(x)^2 = \frac{1-x^2}{n^2}\, T_n'(x)^2\,.$$

and from first principles derive that if q is a polynomial of degree $n-1$ that satisfies

$$|q(x)| \leqslant \frac{n}{\sqrt{1-x^2}} \qquad \text{at the } n \text{ zeros of } T_n\,,$$

then

$$|q(x)| \leqslant |T_n'(x)|\,, \qquad |x| \geqslant \cos\frac{\pi}{2n}\,.$$

(b) Using (a) and the Bernstein inequality

$$|p(x)| \leqslant 1 \quad \Rightarrow \quad |p'(x)| \leqslant \frac{n}{\sqrt{1-x^2}}$$

derive the Markov inequality

$$|p(x)| \leqslant 1 \quad \Rightarrow \quad |p'(x)| \leqslant T_n'(1)\,.$$

4

For a knot sequence $(t_i)_{i=1}^{n+k} \subset [a,b]$ with distinct knots, let

$$M_i(t) := k\,[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}\,, \qquad N_i(t) := (t_{i+k}-t_i)[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}$$

be the sequences of L_1 and L_∞-normalized B-splines, respectively.

(a) Prove that M_i are piecewise-polynomial functions of degree $k-1$ and global smoothness C^{k-2}, with knots $(t_i, \ldots, t_{i+k})$ and with the finite support $[t_i, t_{i+k}]$.

(b) Using the Leibnitz rule for the divided differences, or otherwise, derive the recurrence formula for B-splines:

$$N_{i,k}(t) = \frac{t-t_i}{t_{i+k-1}-t_i}\, N_{i,k-1} + \frac{t_{i+k}-t}{t_{i+k}-t_{i+1}}\, N_{i+1,k-1}\,,$$

where $N_{i,m}$ is the L_∞-normalized B-spline of order m with support $[t_i, t_{i+m}]$.

5

(1) Let $\mathcal{S}_k(\Delta)$ be the space of splines of degree $k-1$ spanned by the B-splines $(N_j)_{j=1}^n$ on a knot sequence $\Delta = (t_j)_{j=1}^{n+k}$ such that $t_j < t_{j+k}$. Let $x = (x_i)_{i=1}^n$ be interpolation points obeying the conditions

$$N_i(x_i) > 0\,,$$

and let $P_x : C[a,b] \to \mathcal{S}_k(\Delta)$ be the map which associates with any $f \in C[a,b]$ the spline $P_x(f)$ from $\mathcal{S}_k$ which interpolates f at (x_i). Prove that

$$\frac{1}{d_k}\,\|A_x^{-1}\|_{\ell_\infty} \leqslant \|P_x\|_{L_\infty} \leqslant \|A_x^{-1}\|_{\ell_\infty}$$

where A_x is the matrix $(N_j(x_i))_{i,j=1}^n$, and d_k is the constant such that

$$\frac{1}{d_k}\,\|a\|_{\ell_\infty} \leqslant \|\sum_{i=1}^n a_i\,N_i\|_{L_\infty} \qquad \forall\, a \in \mathbb{R}^n\,.$$

(2) Consider the case of quadratic interpolating splines on the uniform knot-sequence $(t_1, t_2, \ldots, t_{n+3}) = (1, 2, \ldots, n+3)$ with the interpolating points

$$x_i = t_{i+2} = i+2\,, \qquad i = 1, \ldots, n\,.$$

Prove that $\|P_x\|_{L_\infty} = \mathcal{O}(n)$, hence $P_{\boldsymbol{x}}$ is not bounded uniformly in n. (You may use the equality $d_3 = 3$).

6

(a) Define a multiresolution analysis of $L_2(\mathbb{R})$ with a generator ϕ and explain how it is related to existence of an orthonormal wavelet ψ.

(b) Prove that the following properties of ϕ

$$(1)\ \ \phi(x) = \sum_n a_n \, \phi(2x-n)\,, \qquad (2)\ \ \{\phi(\cdot - n)\} \text{ is orthonormal.}$$

are equivalent to

$$(1')\ \ f(2t) = m(t) f(t)\,, \qquad (2')\ \ \sum |f(t+2\pi k)|^2 \equiv 1 \ \text{ a.e.}$$

where f is the Fourier transform of ϕ, i.e., $f(t) = \int_{\mathbb{R}} \phi(x)\, e^{-ixt}\, dx$.

(c) Verify that conditions (1')-(2') are fulfilled for the function f defined by the following rule:

1. $f(t) = 1\,, \qquad |t| < \frac{2}{3}\pi\,,$
2. $f(t) = 0\,, \qquad |t| \geqslant \frac{4}{3}\pi\,,$
3. $f^2(t) + f^2(t-2\pi) = 1\,, \qquad t \in \left[\frac{2}{3}\pi\,, \frac{4}{3}\pi\right].$

END OF PAPER