UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 3 June, 2010 9:00 am to 12:00 pm

PAPER 61

PLANETARY SYSTEM DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a test particle orbiting a binary comprised of a star and planet of masses $M_\star$ and $M_{\rm pl}$. The binary orbit is circular at semimajor axis $a_{\rm pl}$ with mean motion $n_{\rm pl}$. The particle orbits in the same sense and in the same plane as the binary but at much larger distance. To first order its orbit can be described by two body motion about a mass $M_\star + M_{\rm pl}$ located at the centre of mass of the binary O, with orbital elements a, e, f.

You may assume $r = \dfrac{a(1-e^2)}{(1+e\cos f)}$ and $r^2\dot{f} = \sqrt{G(M_\star + M_{\rm pl})a(1-e^2)}$.

Define a coordinate system centred on O with the x-axis pointing to the location of the planet at $t = 0$, and the y-axis in the direction of the planet's motion at $t = 0$. The orientation of the particle's pericentre with respect to the x-axis is ω. Show that its velocity is

$$\begin{aligned} \dot{x} &= A\left[-\sin(\omega+f) - e\sin\omega\right], \\ \dot{y} &= A\left[\cos(\omega+f) + e\cos\omega\right], \end{aligned} \qquad \text{where} \quad A = \sqrt{\frac{G(M_\star + M_{\rm pl})}{a(1-e^2)}}\,.$$

Show also that the particle's velocity in the astrocentric $x'-y'$ frame with x' parallel to x is

$$\begin{aligned} \dot{x}' &= \dot{x} - \mu\alpha^{-0.5}(1-e^2)^{0.5}A\sin n_{\rm pl}t\,, \\ \dot{y}' &= \dot{y} + \mu\alpha^{-0.5}(1-e^2)^{0.5}A\cos n_{\rm pl}t\,, \end{aligned} \qquad \text{where} \quad \mu = \frac{M_{\rm pl}}{M_\star + M_{\rm pl}} \quad \text{and} \quad \alpha = \frac{a_{\rm pl}}{a}\,.$$

Even though the test particle's orbit is assumed to be fixed, its astrocentric osculating orbital elements (a', e', ω', f') undergo short period variations. Show that

$$\frac{x' - x}{x} = O(\mu\alpha)\,.$$

Derive the astrocentric velocity of the particle v' and show that in the limit that $\mu = 0$ this can be written as $v'^2 = GM_\star(2/r - 1/a)$.

If the particle is on a circular orbit, and $\mu \ll \alpha^{0.5} \ll 1$, find to lowest order an expression for $a'/a - 1$.

Show that to lowest order

$$\begin{aligned} e'\cos(\omega' - \omega - f) &= 2\mu\alpha^{-0.5}\cos(n_{\rm pl}t - \omega - f)\,, \\ e'\sin(\omega' - \omega - f) &= \mu\alpha^{-0.5}\sin(n_{\rm pl}t - \omega - f)\,. \end{aligned}$$

2

A particle orbits a star in an external first order mean motion resonance with a planet that is on a circular orbit in the same plane with mean motion $n_{\rm pl}$. The resonant argument is $\phi = (p+1)\lambda - p\lambda_{\rm pl} - \varpi$, where λ and $\lambda_{\rm pl}$ are the true longitudes of the particle and planet, ϖ is the particle's longitude of pericentre and p is a positive integer. Draw a diagram showing the geometry of the orbit in the frame rotating with the mean motion of the planet for $p = 4$ and $e = 0.2$, indicating the angle ϕ.

Show that radial and tangential forces F_r and F_θ applied to the particle result in a rate of change of angular momentum rF_θ.

Describe how kicks received from conjunctions with the planet in various configurations affect the longitude of subsequent conjunctions.

Lagrange's planetary equations show that the planet's perturbations cause changes to the particle's orbital elements of $\dot{a}_r = -2(p+1)\,Cae\,\sin\phi$, $\dot{e}_r = -C\sin\phi$ and $\dot{\varpi}_r = (C/e)\cos\phi$. The particle is also subject to a dissipative force that damps its eccentricity at a rate $\dot{e}_d = -Ae$. A and C are functions of the particle's semimajor axis. Show that the evolution of the particle's complex eccentricity $z = e\exp(i\phi)$ is given by

$$\dot{z} \approx -iC - (A + 1.5\,pn_{\rm pl}\,xi)z\,,$$

where $x = \dfrac{a-a_r}{a_r} \ll 1$ and a_r is the nominal location of the resonance.

If the dissipative force also causes the particle's semimajor axis to evolve at a rate $\dot{a}_d = -0.8Aa$, show that the location of the fixed points are at

$$e_f = \sqrt{\frac{2}{5(p+1)}}\,, \qquad \text{with} \quad \phi_f = -\psi_f \quad \text{or} \quad \phi_f = \psi_f + \pi\,,$$

where $$\psi_f = \arctan\left(\frac{A}{\sqrt{2.5\,C^2(p+1) - A^2}}\right).$$

Use physical arguments to assess which of the fixed points is stable.

3

A test particle orbits a planet of mass and radius $M_{\rm pl}$ and $R_{\rm pl}$ on a circular orbit that is inclined by I to the planet's equatorial plane (the $x-y$ plane) with a longitude of ascending node Ω relative to the reference direction x. If the particle's mean anomaly is M and its semimajor axis a, find its position in the planetocentric frame in terms of its orbital elements.

The planet is on a circular orbit around a star of mass $M_\star$ at a distance $a_{\rm pl}$ resulting in a disturbing function to the particle's two body planetocentric motion of

$$\mathcal{R} = \frac{GM_\star}{|\boldsymbol{r}_\star - \boldsymbol{r}|} - GM_\star \frac{\boldsymbol{r}\cdot\boldsymbol{r}_\star}{r_\star^3},$$

where $\boldsymbol{r}_\star$ and $\boldsymbol{r}$ are vector offsets of the star and particle from the planet. Show that to lowest order in $a/a_{\rm pl}$ the disturbing function is

$$\mathcal{R} = GM_\star \left(\frac{a^2}{a_{\rm pl}^3}\right)(3\cos^2\psi - 1)/2,$$

where ψ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}_\star$.

The planet's orbital plane is defined by $I_{\rm pl}$ and $\Omega_{\rm pl}$, the inclination and longitude of ascending node relative to the (equatorial) planetocentric frame. Expand $\cos\psi$ in terms of orbital elements up to second order in inclinations.

Show that the value of $(3\cos^2\psi - 1)/2$ averaged over the mean longitudes of the orbits of both the particle and the planet is

$$\langle\langle\, 1.5\cos^2\psi - 0.5\,\rangle\rangle = (1/4) + (3/4)\, I I_{\rm pl}\, \cos(\Omega_{\rm pl} - \Omega) - (3/8)(I^2 + I_{\rm pl}^2).$$

What is the disturbing function for secular perturbations to the particle's orbit due to the star?

The particle is also subject to perturbations due to the planet's oblateness that can be modelled by the disturbing function

$$\mathcal{R} = GM_{\rm pl}\, J_2 \left(\frac{R_{\rm pl}^2}{a^3}\right)(2 - 3I^2)/4,$$

where J_2 is the planet's second gravitational moment. Given Lagrange's planetary equation $\dot{\Omega} = (na^2 I)^{-1}\,\partial\mathcal{R}/\partial I$, where n is the particle's mean motion, and assuming the planet's equatorial plane is aligned with its orbital plane, find the distance from the planet where the rate of precession of the particle's orbital plane due to the planet is equal to that due to the star.

4

The dispersal threshold (the critical specific incident energy for catastrophic disruption) for planetesimals of diameter D is given by $Q^\star_{\rm D} = Q_a D^{-a} + Q_b D^b$, where a and b are positive. Show that the weakest planetesimal is that of size

$$D_{\rm w} \;=\; \left(\frac{a\,Q_a}{b\,Q_b}\right)^{\frac{1}{a+b}} .$$

Find an expression for the size of the smallest planetesimal $D_{\rm wc}$ that would cause catastrophic disruption of the weakest planetesimal if $v_{\rm rel}$ is the relative velocity of collisions.

For $a = 1/2$ and $b = 3/2$, show that

$$D_{\rm wc} \;=\; 2 \cdot 3^{-3/4}\, Q_a^{3/4}\, Q_b^{-5/12}\, v_{\rm rel}^{-2/3} .$$

A planetesimal of mass M and density ρ at a distance r_0 from the star is destroyed in a collision that results in half of its mass ending up in dust on nearly circular orbits, with a size distribution $n(D) = KD^{-\alpha}$ in the range $D_{\rm min}$ to $D_{\rm max} \gg D_{\rm min}$. Find K given that $\alpha < 4$.

The dust evolves due to Poynting-Robertson drag so that $\dot{r} = -\gamma(rD)^{-1}$. Find the time $t_p(D)$ it takes for dust of size D to reach a planet located at $r_p < r_0$.

Discuss the different possible evolutionary paths of the dust as it passes the planet, and how these might depend on the planet's mass and size, and on dust size.

Show that if the planet accretes all of the dust that passes it, its accretion rate at a time t is given by

$$\dot{M}_p \;=\; (2 - \alpha/2)\left[\frac{M}{t_p(D_{\rm max})}\right]\left[\frac{t}{t_p(D_{\rm max})}\right]^{3-\alpha} ,$$

and give the range of time over which this is valid.

Find an expression for the mass surface density of the dust distribution as a function of $\dfrac{t}{t_r(D_{\rm max})}$, where $t_r(D_{\rm max})$ is the time for particles of size $D_{\rm max}$ to reach a radius r.

END OF PAPER