UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 27 May, 2010 1:30 pm to 4:30 pm

PAPER 6

INTRODUCTION TO FUNCTIONAL ANALYSIS

Attempt no more than ***THREE*** *questions,*
and not more than ***TWO*** *from either Section.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1

Prove the geometric Hahn-Banach theorem in the following form. Suppose that A is a non-empty radially open convex subset of a real vector space V, and that U is a linear subspace of V disjoint from A. Show that there exists a hyperplane H such that $U \subseteq H$ and $H \cap A = \emptyset$.

Suppose that f is a non-zero convex real-valued function on a real vector space V with $f(0) = 0$. By considering the set

$$A = \{(x, \beta) \in V \times \mathbf{R} : f(x) < \beta\}$$

or otherwise, show that there exists a linear functional l on V such that $l(x) \leqslant f(x)$ for all $x \in V$.

Now suppose that $(V, \|.\|)$ is a normed space and that f is continuous on V. Show that the null-space of l is closed. Explain why this implies that l is continuous.

2

Which of the following statements are true, and which false? If a statement is true, prove it, and if not, provide a counterexample.

(i) If a separable compact Hausdorff topological space (X, τ) has a dense subset Y which is metrizable in the subspace topology then (X, τ) is metrizable.

(ii) If the weak topology and the norm topology are the same on the unit ball B of a normed space $(E, \|.\|)$ then E is finite-dimensional.

(iii) If $(E, \|.\|)$ is a separable normed space, then its unit ball B is metrizable in the weak topology.

(iv) If $(E, \|.\|)$ is a separable normed space, then the unit ball B' of its dual E' is metrizable in the weak* topology.

3

Prove the existence of Haar measure on a compact Hausdorff group G.

[*You may assume Hall's marriage theorem, but should establish properties of minimal left-V nets that you need.*]

SECTION II

4

What is a *uniform algebra* A on a compact Hausdorff space K? Explain why there is a homeomorphism δ of K onto a closed subset $j(K)$ of the carrier space Φ_A.

A uniform algebra A is said to be *natural* if $j(K) = \Phi_A$. Show that the following are equivalent:

(i) A is natural;

(ii) If $f_1, \ldots , f_n \in A$ and $\cap_{j=1}^{n}\{x \in K : f_j(x) = 0\} = \emptyset$ then there exist $g_1, \ldots , g_n$ such that $f_1 g_1 + \cdots + f_n g_n = 1$.

[*Hint: Consider the ideal generated by* $f_1, \ldots, f_n$.]

Use this result to show that $C(K)$ is natural.

Show that the disc algebra $A(\bar{\mathbf{D}})$ is natural.

Let $C = \{f \in C(\bar{\mathbf{D}}) :$ there exists $g \in A(\bar{\mathbf{D}})$ such that $f(z) = g(z)$ for $|z| = 1\}$. Determine the carrier space Φ_C of C.

5

What is the *spectrum* $\sigma(a)$ of an element a of a unital Banach algebra A? State the relation between $\sigma(ba)$ and $\sigma(ab)$. Give an example where $\sigma(ba) \neq \sigma(ab)$.

What is a *positive* element of a unital C^*-algebra A? Explain, with examples, how the Gelfand functional calculus can be used to obtain properties of positive elements.

Show that if a and b are positive then $a + b$ is positive.

Show that if $a \in A$ then a^*a is positive.

Show that if a is positive and $c \in A$ then c^*ac is positive.

As usual, write $a \leqslant b$ if $b - a$ is positive. Suppose that $1 \leqslant a \leqslant b$.

(i) Show that a and b are invertible, and that $0 < b^{-1} \leqslant a^{-1} \leqslant 1$.

(ii) Show that $\sigma(ab) \subseteq \mathbf{R}^+$.

(iii) Give an example where $b^2 - a^2$ is not positive.

END OF PAPER