UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 28 May, 2010 1:30 pm to 4:30 pm

PAPER 59

ASTROPHYSICAL FLUID DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + u\cdot\nabla\rho &= -\rho\nabla\cdot u\,, \\
\frac{\partial p}{\partial t} + u\cdot\nabla p &= -\gamma p\nabla\cdot u\,, \\
\rho\left(\frac{\partial u}{\partial t} + u\cdot\nabla u\right) &= -\rho\nabla\Phi - \nabla p + \frac{1}{\mu_0}(\nabla\times B)\times B\,, \\
\frac{\partial B}{\partial t} &= \nabla\times(u\times B)\,, \\
\nabla\cdot B &= 0\,, \\
\nabla^2\Phi &= 4\pi G\rho\,.
\end{aligned}$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Small diffusionless disturbances (velocity $\mathbf{u}$, magnetic field $\mathbf{b}$) occur to a motionless state with a magnetic field $\mathbf{B} = Jr\hat{\mathbf{e}}_\phi$ (so that $\nabla \times \mathbf{B} = 2J\hat{\mathbf{e}}_z$) in cylindrical polar coordinates (r, ϕ, z). The flow may be assumed to be solenoidal.

Consider disturbances proportional to $e^{im\phi + ikz + i\omega t}$, and show that they obey the equations

$$i\omega\mathbf{u} = -\nabla\tilde{p} + \frac{1}{\mu_0\rho}\left(imJ\mathbf{b} + 2J\hat{\mathbf{e}}_z \times \mathbf{b}\right),$$
$$i\omega\mathbf{b} = imJ\mathbf{u},$$

where $\tilde{p}$ is a modified pressure and ρ is the fluid density. By eliminating the velocity and pressure show that

$$\nabla^2\mathbf{b} = -\frac{4\,k^2m^2\tilde{J}^4}{(\omega^2 - m^2\tilde{J}^2)^2}\,\mathbf{b},$$

where $\tilde{J} = J/\sqrt{\mu_0\rho}$.

Consider eigenmodes for which $\nabla^2\mathbf{b} = -(\alpha^2 + k^2)\mathbf{b}$, where α is real. Show that ω^2 is real and that $\omega^2 > 0$ for $|m| \geqslant 2$. Give conditions on m, k and α for which there is instability when $|m| = 1$.

2

Write down the equations of steady, spherical accretion of a barotropic gas (with $p = p(\rho)$) in an arbitrary gravitational potential $\Phi(R)$. It may be assumed that the gas is uniform and at rest at infinity with pressure p_0.

Explain the concept of the sonic radius R_s. Argue that the accretion flow, which is subsonic at infinity and supersonic for sufficiently small R, passes through a sonic point if one exists.

Now suppose the gas is polytropic, with $p = K\rho^{1+1/m}$. Determine the values of m for which a sonic point exists in the case $\Phi = -\lambda R^{-2} - \mu R^{-1}$, and give the associated accretion rate.

3

Consider a spherically symmetric self-gravitating star V in hydrostatic equilibrium. It may be assumed that the pressure p and density ρ both vanish at the surface $R = a$. Making standard assumptions, derive the equations governing small displacements $\boldsymbol{\xi}$, and derive the equation for the frequency ω:

$$\omega^2 \int_V \rho |\boldsymbol{\xi}|^2 \, dV \; = \; -\frac{1}{4\pi G} \int_\infty |\nabla \delta\phi|^2 \; dV + \int_V \left(\frac{|\delta p|^2}{\gamma\rho} + \rho N^2 |\xi_R|^2 \right) dV \, ,$$

where $\delta p, \delta\phi$ are the Lagrangian displacements of p, ϕ, respectively, γ is the ratio of specific heats, ξ_R is the radial component of $\boldsymbol{\xi}$, and $N^2 = g\left(\frac{1}{\gamma}\frac{d\ln p}{dR} - \frac{d\ln\rho}{dR}\right)$ is the buoyancy frequency.

Making any plausible assumptions about the density and pressure distributions, estimate the importance of the first two terms on the right hand side of this expression. Show that the first term (self-gravity) can be neglected for sufficiently small scales of disturbance.

In this case, and if $N^2 < 0$ for $c < R < b < a$, give an explicit form of $\boldsymbol{\xi}$ that yields a negative value of ω^2, and hence demonstrate instability.

4

Write an essay on the Magneto-Rotational Instability. You should consider only axisymmetric instabilities. Your essay should cover both the basic instability mechanism for a uniform field, and any new effects that arise for more general fields.

END OF PAPER