UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2010 1:30 pm to 4:30 pm

PAPER 58

STRUCTURE AND EVOLUTION OF STARS

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

You may use the equations and results given below without proof.

The symbols used in these equations have the meanings that were given in lectures. Candidates are reminded of the equations of stellar structure in the form:

$$\frac{dm}{dr} = 4\pi r^2 \rho\,, \qquad \frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,, \qquad \frac{dL_{\rm r}}{dr} = 4\pi r^2 \rho\epsilon\,.$$

In a radiative region

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_{\rm r}}{16\pi a c r^2 T^3}\,.$$

In a convective region

$$\frac{dT}{dr} = \frac{(\Gamma_2 - 1)T}{\Gamma_2 P}\frac{dP}{dr}\,.$$

The luminosity, radius and effective temperature are related by $L = 4\pi R^2 \sigma T_{\rm e}^4\,.$

The equation of state for an ideal gas and radiation is $P = \dfrac{\mathcal{R}\rho\mathcal{T}}{\mu} + \dfrac{aT^3}{3}\,,$
with $1/\mu = 2X + 3Y/4 + Z/2\,.$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a cluster of chemically homogeneous massive stars. The stellar material is an ideal gas and radiation pressure is negligible. The energy generation, opacity and whether the stellar interiors are convective or radiative depends on a star's mass. There are three mass-ranges of behaviour over the full main sequence:

(i) For $M/M_{\odot} > 2$, the stars are fully radiative, energy generation is by the CNO cycle and opacity dominated by electron-scattering, such that $\epsilon = \epsilon_0 X \rho T^{13}$ and $\kappa = \kappa_0$;

(ii) For $0.5 < M/M_{\odot} < 2$, the stars are also fully radiative, energy generation is by the p-p chain and opacity determined by Kramers' opacity, such that $\epsilon = \epsilon_0 X \rho T^7$ and $\kappa = \kappa_0 \rho T^{-7/2}$;

(iii) For $M < 0.5 M_{\odot}$, the stars are fully convective, energy generation is by the p-p chain and opacity dominated by H^- anions, such that $\epsilon = \epsilon_0 X \rho T^3$ and $\kappa = \kappa_0 \rho^{\frac{1}{2}} T^9$.

Here κ_0 and ϵ_0 are constants. Using homology show for the three different types of main-sequence star that $L \propto M^{\alpha}$ with $\alpha = 3$, $5\,\frac{6}{19}$ and $1\,\frac{47}{66}$ respectively. Then determine the gradient that each main sequence makes in the theoretical Hertzsprung-Russell diagram and sketch this diagram showing the three main sequences and how they combine.

For case (iii), when the stars are convective, show for a star of fixed mass that $L \propto X^{17/66} \mu^{103/66}$, assuming that $Z = 0$ and initially $X = 1$. Using this result, estimate how such a star evolves away from the main sequence as it burns hydrogen to helium. Sketch an example track onto your HR diagram. Why can this analysis not be performed for the stars that have radiative interiors?

2

A white dwarf has a helium core of mass M and radius R and a thin hydrogen-rich, non-degenerate envelope of mass $M_{\rm env} \ll M$ and thickness $H \ll R$. Writing $z = r - R$, the surface density $\Sigma(z)$ in the envelope is defined by $d\Sigma = -\rho dz$ with $\Sigma = 0$ at $z = H$ and $\Sigma = \Sigma_o = M_{env}/4\pi R^2$ at $z = 0$. Show that, to a good approximation, in the envelope $P = \Sigma g$ where $g = GM/R^2$.

Show that the approximate equations for the structure of the envelope are

$$\frac{dF}{d\Sigma} = -\epsilon \qquad \text{and} \qquad \frac{dT}{d\Sigma} = \frac{3\,\kappa F}{4\,acT^3}\,,$$

where $F = L_r/4\,\pi R^2$.

The opacity is of the form $\kappa = \kappa_0 \rho T^{-3}$ where κ_0 is constant. The white dwarf's luminosity is produced solely by hydrogen burning in the envelope. The hydrogen burns steadily, mainly at the base of the envelope with $\epsilon = \epsilon_0 \rho T^{25}$, where ϵ_0 is a constant. Radiation pressure can be neglected. Setting $y = T^8$ and $x = \frac{1}{2}\,\Sigma^2$ show that the structure equations can be written in the form

$$\frac{d^2y}{dx^2} = -\omega^2 y^3\,,$$

where ω^2 is a positive constant.

Show that the appropriate boundary conditions are: at $x = 0$, $y = 0$ and $\frac{dy}{dx} = \frac{AL}{4\pi R^2}$, where A is a constant; at $x = \frac{1}{2}\,\Sigma_0^2$, $y = T_0^8$ and $\frac{dy}{dx} = 0$.

Multiply both sides by $\frac{dy}{dx}$, integrate and use the boundary conditions to show that

$$T_0 \propto \left(\frac{L}{4\,\pi R^2}\right)^{1/16}.$$

Then integrate again and show that

$$\frac{L}{4\,\pi R^2} \propto \left(\frac{M_{\rm env}}{4\,\pi R^2}\right)^{-4}.$$

Comment on the stability of hydrogen burning in a white dwarf envelope and what might happen as a white dwarf accretes hydrogen material from a companion donor star.

3

Write brief notes on **four** of the following topics:

(i) Polytropes and their use as stellar models.

(ii) The lightcurves of supernovae.

(iii) Evidence that nuclear reactions occur within stars.

(iv) Thermal pulses in asymptotic giant branch stars.

(v) The Schwarzschild criterion for convection.

4

Consider a semi-detached binary, with conservative Roche lobe overflow. Show that the radius a of the circular orbit satisfies

$$a \propto \frac{(1+q)^4}{q^2} \,,$$

where $q = M_1/M_2$ is the mass ratio of the two components (loser/gainer).

In a certain range of mass ratios, the radius of the Roche lobe around the loser can be approximated by

$$R_{\rm L} \approx 0.4\, a q^{2/9} \,.$$

Show that as the loser, of mass M_1, conservatively transfers mass to its companion, its Roche-lobe radius changes at a rate

$$\frac{d \log_e R_{\rm L}}{dt} = \alpha \frac{d \log_e M_1}{dt} \,,$$

where

$$\alpha = \frac{20}{9}\left(q - \frac{4}{5}\right) \,.$$

A star of mass M_1 and age t has a radius R_1 which changes in response to internal nuclear evolution and to variation in mass (provided the variation is slow), according to

$$\log_e R_1 = \beta \log_e M_1 + t/t_{\rm nuc} \,,$$

where β, the slope of the ZAMS radius-mass relation, and $t_{\rm nuc}$, the nuclear timescale, can be taken as constants. As long as $R_1 < R_{\rm L}$ the mass M_1 remains constant but once $R_1 > R_{\rm L}$ mass starts to flow at a rate given by

$$\frac{d \log_e M_1}{dt} = -\frac{1}{t_{\rm dyn}} \log_e \frac{R_1}{R_{\rm L}} \,,$$

where $t_{\rm dyn}$ is the dynamical timescale, also constant $(t_{\rm dyn} \ll t_{\rm nuc})$.

Let $f = \log_e (R_1/R_{\rm L})$. As long as f is negative show that it satisfies the differential equation

$$\frac{df}{dt} = \frac{1}{t_{\rm nuc}} \,,$$

and find a corresponding first-order linear differential equation satisfied by f when it is positive. Show that as long as $\beta > \alpha$, f tends to a small constant positive value,

$$f \to \frac{1}{\beta - \alpha} \frac{t_{\rm dyn}}{t_{\rm nuc}} \,,$$

implying steady mass transfer on a nuclear timescale; but that if $\beta < \alpha$, f grows exponentially on a dynamical timescale.

On the lower main sequence $\beta \approx 1$, while on the upper main sequence $\beta \approx 0.5$. Find the corresponding ranges of initial mass ratio q_0 for which mass transfer can proceed steadily on a nuclear timescale once the primary has filled its Roche lobe.

Comment on how your results would change if mass transfer was not conservative. Also speculate what would happen to the binary if the secondary star began to fill its Roche lobe after accreting material from the donor star.

END OF PAPER