UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 2 June, 2010 1:30 pm to 4:30 pm

PAPER 57

ASTROPHYSICAL DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) A self-gravitating stellar system has a power-law density profile

$$\rho = \rho_0 \left(\frac{r_0}{r}\right)^\alpha ,$$

where ρ_0 and r_0 are the length and density scales, while α is a constant satisfying $1 < \alpha < 3$. Show that the gravitational potential is

$$\phi = \frac{4\pi G \rho_0 r_0^2}{(3-\alpha)(2-\alpha)} \left(\frac{r}{r_0}\right)^{2-\alpha} .$$

What does this formula become when $\alpha = 2$?

The Jeans equation for an isotropic spherical system reads

$$\frac{d}{dr}\left(\rho\langle v_r^2\rangle\right) = -\rho\,\frac{d\phi}{dr} .$$

Show that radial velocity dispersion $\langle v_r^2\rangle$ is

$$\langle v_r^2\rangle = \frac{2\pi G \rho_0 r_0^2}{(3-\alpha)(\alpha-1)} \left(\frac{r}{r_0}\right)^{2-\alpha} .$$

(ii) Suppose a stellar system has the phase space distribution function

$$f(E) = F_0\, E^{-n-3/2} ,$$

where F_0 is a constant, E is the energy and the potential ϕ is zero at the centre of the system. Show that the density ρ satisfies

$$\rho = \rho_n\, \phi^{-n} ,$$

where ρ_n is a constant. What values of n are permitted?

Show that the dimensionless radius $s = r/b$ and the potential $\psi = \phi/\phi_0$ satisfy

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi}{ds}\right) = 3\,\psi^{-n} ,$$

where ϕ_0 is arbitrary and $b = (\frac{4}{3}\pi G\, \phi_0^{-n-1}\rho_n)^{-1/2}$.

Hence, show that the equations admit power-law solutions

$$\rho \propto r^{-\alpha}, \qquad \text{for } 0 < \alpha \leqslant 2 .$$

Why does this method not find the power-law solutions with $2 < \alpha < 3$ found in part (i)? Why does the method in part (i) not find the power-law solutions with $0 < \alpha \leqslant 1$?

[*Hint: You may assume without proof the standard integral*

$$4\int_0^{\pi/2} \sin^2\theta\, \cos^{2n-1}\theta\, d\theta = \frac{\sqrt{\pi}\,\Gamma(n)}{\Gamma(n+3/2)} \qquad \textit{valid for } n > 0 .]$$

2

(i) A self-gravitating ideal gas with density $\rho(r)$ and potential $\phi(r)$ is contained within a spherical box of radius $r_{\rm b}$. Suppose that the gas is thermally conducting so that the equilibrium state is isothermal. Show that the total energy E of the gas is

$$E = \frac{3}{2} N k_{\rm B} T + 2\pi \int_0^{r_{\rm b}} r^2 \rho(r)\, \phi(r)\, dr\,,$$

where $k_{\rm B}$ is Boltzmann's constant, T is the temperature and N the number of gas molecules.

Write down the distribution function of the gas, and derive a differential equation for the density ρ.

Suppose the container is placed in contact with a heat bath initially maintained at a constant, very high temperature. Draw a graph showing the behaviour of the temperature and energy of the gas as the temperature of the heat bath is gradually reduced.

Mark on the graph the point at which the heat capacity first becomes negative, and the point corresponding to the onset of the gravothermal catastrophe.

(ii) Consider a rigid sphere of radius $r_{\rm max}$. Inside this sphere and concentric with it is a second sphere whose radius can vary between $r_{\rm min}$ and $r_{\rm max}$ which contains an ideal gas. The space between the two spheres is empty. The inner sphere has a potential $-b/r$, which tends to make it collapse, but the collapse is opposed by the pressure P of the gas. Show that the equilibrium radius of the sphere is

$$r = \left(\frac{b}{4\pi P}\right)^{1/4}, \qquad r_{\rm min} < r < r_{\rm max}\,.$$

If the gas is ideal and monatomic, show that

$$r = \frac{b}{2\,aT}\,, \qquad r_{\rm min} < r < r_{\rm max}\,,$$

where $a = 3\,N k_{\rm B}/2\,.$

Hence, show that that the kinetic energy of the gas is aT and that the energy of the system is E where

$$E = aT - \frac{b}{r}\,,$$

and demonstrate that the heat capacity is negative.

Interpret your result physically.

3

Let (r, θ, ϕ) be spherical polar coordinates and consider the gravitational potential

$$\phi(r, \theta) \;=\; v_0^2 \, \log\bigl(r + r\,|\cos\theta|\bigr)\,,$$

where v_0 is a constant. Show that ϕ satisfies Laplace's equation in the half-spaces $z > 0$ and $z < 0$.

Use Gauss's Theorem to show that there is an infinitesimally thin disk (the *Mestel disk*) occupying the plane $z = 0$, and find its surface density $\Sigma(r)$.

Show that a star in the Mestel disk conserves its energy E and its angular momentum component L_z. Hence, explain why the phase-space distribution function F of the Mestel disk depends on E and L_z only.

By integrating over velocity space, show that the Mestel disk has a one-parameter family of distribution functions of form

$$F(E, L_z) \;=\; F\,|L_z|^q \exp\bigl(-E/\sigma_0^2\bigr)\,,$$

for suitable choices of the constants F, q and σ_0.

Find the radial and tangential velocity dispersion in the disk. To what do the limits $q \to \infty$ and $q \to -1$ correspond?

[*Hint: You are reminded of the standard integral*

$$\int_{-\infty}^{\infty} \exp(-\alpha v^2)\, dv \;=\; \sqrt{\frac{\pi}{\alpha}}\,,$$

as well as the definition of the Gamma function as

$$\Gamma(q+1) \;=\; \int_0^{\infty} x^q \, \exp(-x)\, dx\,.]$$

4

(i) Let (x, y, z) be Cartesian coordinates. Suppose the potential in the plane $z = 0$ of the core of a triaxial galaxy can be approximated by

$$\phi = \frac{1}{2}\,\omega^2\left(x^2 + 4\,y^2\right),$$

where ω is a constant. By introducing canonical momenta p_x and p_y and separating the Hamilton-Jacobi equation, show that the energies

$$E_x = \frac{1}{2}\left(p_x^2 + \omega^2\,x^2\right), \qquad E_y = \frac{1}{2}\left(p_y^2 + 4\,\omega^2\,y^2\right),$$

are integrals of the motion.

By integrating the equations of motion, or otherwise, show that there is an additional integral J given by

$$J = 2\arctan\left(\frac{p_x}{\omega x}\right) - \arctan\left(\frac{p_y}{2\,\omega y}\right),$$

What does this imply about motion in the (x, y) plane?

(ii) Define *action-angle coordinates.* Write down Hamilton's equations in action-angle coordinates.

What is meant by the term *adiabatic invariant*?

Suppose a star is moving on a bound orbit of energy E_0 and angular momentum L_0 in the potential

$$\phi = -\frac{\gamma_0}{r^n}, \qquad 0 < n < 2,$$

where γ_0 and n are constants. Determine the change in energy and angular momentum if γ_0 changes slowly and adiabatically to γ_1.

Discuss what happens to (a) a star moving on a circular orbit of initial radius r_0 and (b) a star moving on a radial orbit with initial apocentre r_a.

END OF PAPER