UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 3 June, 2010 1:30 pm to 4:30 pm

PAPER 55

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The action for the gravitational field in three spacetime dimensions can be written as

$$I = \int \quad (R^{ab} \wedge E^c + \lambda E^a \wedge E^b \wedge E^c)\epsilon_{abc},$$

where ϵ_{abc} is the three dimensional alternating symbol, E^a is an orthonormal basis of 1-forms, R^{ab} is the curvature 2-form of some connection 1-form ω_{ab} and λ is a real constant.

Find the two equations of motion of this theory.

Suppose that the line element is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2,$$

where t is a time co-ordinate, r is a radial co-ordinate and θ an angular co-ordinate. Determine $f(r)$ given that this line element satisfies the equations of motion for this theory, and $f \to 1$ as $r \to 0$.

2

The action for Yang-Mills theory in four spacetime dimensions is given

$$I = \int tr\,(*F \wedge F)$$

where F is a Lie-algebra valued 2-form field strength constructed from a Lie-algebra valued 1-form potential A.

Find the equations of motion.

Suppose one performs an infinitesimal gauge transformation parametrized by η. Find the gauge variations δA and δF of A and F respectively.

Show that the action I is gauge invariant, up to possible boundary terms which should be evaluated if present.

Suppose that one adds to I the term

$$\int tr\,(F \wedge F).$$

Show that this term is similarly gauge invariant.

Does the addition of this term to the action affect the equations of motion?

3

Describe how to calculate the zeta function for the operator $-\Box$ in a compact space without boundary and with a positive definite metric g_{ab}.

In dimension two, find an expression for $\zeta(0)$ in terms of the curvature.

4

A spinor field ψ propagates in a spacetime with metric g_{ab}.

Describe how to construct the vierbein from g_{ab} and explain how local Lorentz transformations act on components of the vierbein.

Explain carefully how ψ transforms under local Lorentz transformations.

Find an expression for the covariant derivative $\nabla_a \psi$ of ψ giving a careful justification for your answer.

Evaluate $\nabla_{[a}\nabla_{b]}\psi$.

END OF PAPER