UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Friday, 4 June, 2010 1:30 pm to 4:30 pm

## PAPER 54

## BLACK HOLES

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

(a)(i) Let $V^a$ be a Killing vector field. Prove that

$$\nabla_a \nabla_b V^c = R^c{}_{bad} V^d .$$

(ii) Let $V^a$ be a Killing vector field in a vacuum spacetime. Show that $F_{ab} = 2\partial_{[a} V_{b]}$ satisfies Maxwell's equations.

(iii) Consider Minkowski spacetime in spherical polar coordinates $(t, r, \theta, \phi)$. What is the interpretation of the solution $F_{ab}$ for $V = \partial/\partial\phi$ ?
[*Hint: transform* $V_a$ *to Cartesian coordinates.*]

(iv) What is the interpretation of $F_{ab}$ for $V = \partial/\partial t$ and $V = \partial/\partial\phi$ in the Schwarzschild spacetime (with $M > 0$)?

(b) The following metric solves Einstein's equation in 3 dimensions with a negative cosmological constant:

$$ds^2 = -\left(\frac{r^2}{\ell^2} - M\right) dt^2 + \frac{dr^2}{(r^2/\ell^2 - M)} + r^2\, d\phi^2 ,$$

where $\phi \sim \phi + 2\pi$, and $\ell$, $M$ are constants with $\ell > 0$.

(i) For $M > 0$, show that this spacetime can be analytically extended such that the resulting spacetime has a Killing horizon at $r = \ell\sqrt{M}$, where the associated Killing vector field coincides with $\partial/\partial t$ for $r > \ell\sqrt{M}$.

(ii) Determine the surface gravity of this Killing horizon.

(iii) For $M < 0$, show that the spacetime is nakedly singular unless $M$ takes a certain value.

**2**

The *Vaidya spacetime* has metric

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2\, dv\, dr + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) ,$$

where $m(v)$ is an arbitrary smooth positive function. It is a solution of Einstein's equation for an energy-momentum tensor with only one non-vanishing component:

$$T_{vv} = \frac{\dot{m}(v)}{4\pi\, r^2} ,$$

where a dot denotes a derivative with respect to $v$.

(a) Show that curves of constant $v, \theta, \phi$ are null geodesics.

(b) What condition(s) must $m(v)$ obey for the dominant energy condition to be satisfied? (*Hint: note that $-\partial/\partial r$ is future-directed because it is tangent to ingoing radial null geodesics.*)

(c) Assume that this spacetime contains an event horizon. By spherical symmetry, this must have equation $r = r(v)$. Show that $r(v)$ satisfies

$$2r(v)\, \dot{r}(v) = r(v) - 2m(v) .$$

(d) Let $m(v) = M_0$ for $v < 0$ and $m(v) = M_1 > M_0$ for $v > v_0$, where $M_0$ and $M_1$ are positive constants, with $m(v)$ monotonically increasing for $0 < v < v_0$. Explain briefly, using a Penrose diagram, why $r(v) = 2M_1$ for $v > v_0$. (You may assume standard properties of the Schwarzschild black hole.)

(e) Now assume that $M_1 - M_0$ is small. Write $m(v) = M_0 + \mu(v)$, where $\mu(v)$ is small, with $\mu(v) = 0$ for $v < 0$ and $\mu(v) = M_1 - M_0$ for $v > v_0$. Write $r(v) = 2M_0 + \rho(v)$, where $\rho(v)$ is small. Working to first order, solve the above equation to determine $\rho(v)$. Hence show that $r(v) \to 2M_0$ as $v \to -\infty$.

(f) Still assuming $M_1 - M_0$ to be small, calculate the total energy of the matter that crosses the horizon. Hence show that the first law of black hole mechanics is satisfied. (*Hint: the energy-momentum tensor is small so one can treat the background as exactly Schwarzschild when calculating the energy that crosses the horizon.*)

**3**

(a) Show that the normal to a null hypersurface is tangent to null geodesics within the surface.

(b) Define the *expansion*, *shear* and *twist* of a null geodesic congruence.

(c) Derive Raychaudhuri's equation.

(d) Prove that the expansion, shear and twist of the generators of a Killing horizon must vanish.

**4**

Write an essay explaining Hawking's argument that black holes must emit thermal radiation.

## END OF PAPER