UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 4 June, 2010 1:30 pm to 4:30 pm

PAPER 54

BLACK HOLES

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a)(i) Let V^a be a Killing vector field. Prove that

$$\nabla_a \nabla_b V^c = R^c{}_{bad} \, V^d \, .$$

(ii) Let V^a be a Killing vector field in a vacuum spacetime. Show that $F_{ab} = 2\partial_{[a} V_{b]}$ satisfies Maxwell's equations.

(iii) Consider Minkowski spacetime in spherical polar coordinates (t, r, θ, ϕ). What is the interpretation of the solution F_{ab} for $V = \partial/\partial\phi$?
[*Hint: transform* V_a *to Cartesian coordinates.*]

(iv) What is the interpretation of F_{ab} for $V = \partial/\partial t$ and $V = \partial/\partial\phi$ in the Schwarzschild spacetime (with $M > 0$)?

(b) The following metric solves Einstein's equation in 3 dimensions with a negative cosmological constant:

$$ds^2 = -\left(\frac{r^2}{\ell^2} - M\right) dt^2 + \frac{dr^2}{(r^2/\ell^2 - M)} + r^2 \, d\phi^2 \, ,$$

where $\phi \sim \phi + 2\pi$, and ℓ, M are constants with $\ell > 0$.

(i) For $M > 0$, show that this spacetime can be analytically extended such that the resulting spacetime has a Killing horizon at $r = \ell\sqrt{M}$, where the associated Killing vector field coincides with $\partial/\partial t$ for $r > \ell\sqrt{M}$.

(ii) Determine the surface gravity of this Killing horizon.

(iii) For $M < 0$, show that the spacetime is nakedly singular unless M takes a certain value.

2

The *Vaidya spacetime* has metric

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2\, dv\, dr + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) ,$$

where $m(v)$ is an arbitrary smooth positive function. It is a solution of Einstein's equation for an energy-momentum tensor with only one non-vanishing component:

$$T_{vv} = \frac{\dot{m}(v)}{4\pi\, r^2} ,$$

where a dot denotes a derivative with respect to v.

(a) Show that curves of constant v, θ, ϕ are null geodesics.

(b) What condition(s) must $m(v)$ obey for the dominant energy condition to be satisfied? (*Hint: note that $-\partial/\partial r$ is future-directed because it is tangent to ingoing radial null geodesics.*)

(c) Assume that this spacetime contains an event horizon. By spherical symmetry, this must have equation $r = r(v)$. Show that $r(v)$ satisfies

$$2r(v)\, \dot{r}(v) = r(v) - 2m(v) .$$

(d) Let $m(v) = M_0$ for $v < 0$ and $m(v) = M_1 > M_0$ for $v > v_0$, where M_0 and M_1 are positive constants, with $m(v)$ monotonically increasing for $0 < v < v_0$. Explain briefly, using a Penrose diagram, why $r(v) = 2M_1$ for $v > v_0$. (You may assume standard properties of the Schwarzschild black hole.)

(e) Now assume that $M_1 - M_0$ is small. Write $m(v) = M_0 + \mu(v)$, where $\mu(v)$ is small, with $\mu(v) = 0$ for $v < 0$ and $\mu(v) = M_1 - M_0$ for $v > v_0$. Write $r(v) = 2M_0 + \rho(v)$, where $\rho(v)$ is small. Working to first order, solve the above equation to determine $\rho(v)$. Hence show that $r(v) \to 2M_0$ as $v \to -\infty$.

(f) Still assuming $M_1 - M_0$ to be small, calculate the total energy of the matter that crosses the horizon. Hence show that the first law of black hole mechanics is satisfied. (*Hint: the energy-momentum tensor is small so one can treat the background as exactly Schwarzschild when calculating the energy that crosses the horizon.*)

3

(a) Show that the normal to a null hypersurface is tangent to null geodesics within the surface.

(b) Define the *expansion*, *shear* and *twist* of a null geodesic congruence.

(c) Derive Raychaudhuri's equation.

(d) Prove that the expansion, shear and twist of the generators of a Killing horizon must vanish.

4

Write an essay explaining Hawking's argument that black holes must emit thermal radiation.

END OF PAPER