UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2010 9:00 am to 12:00 pm

PAPER 53

COSMOLOGY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) The expansion of an isotropic and spatially-homogeneous universe in comoving proper time, t, is described by the equations (where $\dot{a}$ denotes da/dt)

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \qquad \text{and} \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right) + \frac{\Lambda}{3}\,.$$

Give a brief physical interpretation of these equations. Use them to show that

$$\dot{\rho} + 3\,\frac{\dot{a}}{a}\left(\rho + P\right) = 0\,,$$

and explain the physical meaning of this equation.

Define the deceleration parameter, q, and the density parameters Ω_m and Ω_Λ. If the universe contains only matter with zero pressure and $\Lambda > 0$, show that

$$q = \frac{1}{2}\,\Omega_m - \Omega_\Lambda\,.$$

If, today, 30% of the total density of the universe is in the form of matter with zero pressure and the remainder is in the form of dark energy with an equation of state $P = -\rho$, calculate the redshift when the expansion of the universe changes from deceleration to acceleration.

If the spatial geometry of this universe is Euclidean, show that

$$\frac{\dot{a}^2}{a^2} = H_0^2\left(\Omega_{m0}\,a^{-3} + 1 - \Omega_{m0}\right),$$

where H_0 is the Hubble expansion rate today, Ω_{m0} the value of the matter density parameter today, and we normalize $a = a_0 \equiv 1$ today. Defining the time origin by $a(0) = 0$, show that the solution is

$$a(t) = \left(\frac{\Omega_{m0}}{1 - \Omega_{m0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}\,H_0\,t\,\sqrt{1 - \Omega_{m0}}\right).$$

Interpret the limiting behaviour of this solution as $t \to 0$ and $t \to \infty$. Would you expect the inclusion of a negative spatial curvature to affect these limiting behaviours? Explain briefly why this solution provides a good description of the observed expansion of the universe.

(ii) What condition on the density ρ and pressure P is needed for an inflationary cosmology to be able to solve the flatness and horizon problems? Give an example of a type of matter field that could satisfy this requirement and show how the condition for inflation to occur can be met by this field.

Determine whether the following examples of cosmological scale-factor evolution can permit phases of inflationary expansion during the early universe over the time interval $t > 0$:

(a) $a(t) = \ln(Ct)$, $C > 0$ constant;

(b) $a(t) = t^n$, with $n > 0$ constant;

(c) $a(t) = \exp(At^n)$, with constants $A > 0$ and $0 < n \leqslant 1$.

2

(i) Describe the sequence of events that leads to the primordial nucleosynthesis of helium-4 during the radiation-dominated early universe and show how the final abundances of neutrons and protons, and of helium-4, can be estimated using the cosmological expansion time,

$$t = \left(\frac{3}{32\,\pi\, G\rho}\right)^{1/2} = g_*^{-1/2}\left(10^{10}\,\mathrm{K}/T\right)^2 \mathrm{s}\,,$$

and the weak interaction time,

$$t_{\mathrm{wk}} = \left(\frac{\hbar}{G_{\mathrm{weak}}^2\, k_{\mathrm{B}}\, T^5}\right) = \left(10^{10}\,\mathrm{K}/T\right)^5 \mathrm{s}\,,$$

at temperature T, where g_* is the total number of relativistic degrees of freedom present in bosons and fermions at temperature T and k_{B} is Boltzmann's constant.

Explain the effect on the abundance of helium-4 produced by each of the following changes to the standard model you have described:

(a) A reduction in the value of Newton's gravitation constant, G;

(b) A reduction in the neutron half-life;

(c) A fourth neutrino species with mass of 0.001 eV;

(d) An increase in the measured value of the neutron-proton mass difference;

(e) A small change in the baryon density of the universe;

(f) The presence of a cosmological population of 50 GeV weakly interacting neutral particles;

(g) A change in the cosmological metric which leads to faster cosmological expansion rate at a given temperature.

(ii) A relativistic species of hypothetical H particles decouples from the interacting equilibrium sea of particles in the universe at a temperature T_H, which exceeds that equivalent to the muon rest mass (105 MeV) but is less than the pion mass (135 MeV). Calculate the ratio of the temperature of the H particles to that of the photons after the annihilation of muon-antimuon pairs just after the universe cools through 105 MeV. What happens when the temperature of the universe subsequently drops below 0.59 MeV ?

[Assume that the universe contains three generations of light relativistic neutrinos and the entropy density in thermal equilibrium at temperature T, density ρ and pressure P is $s = (\rho + P)T^{-1}$.]

3 Consider linear perturbations to a spatially-flat Robertson-Walker metric:

$$ds^2 = a^2(\eta)\left\{(1+2\psi)d\eta^2 - 2B_i\, dx^i d\eta - [(1-2\phi)\delta_{ij} + 2E_{ij}]\, dx^i dx^j\right\} ,$$

where E_{ij} is symmetric and trace-free.

(i) Under a gauge transformation, $\tilde{\eta} = \eta + T(\eta, x^i)$ and $\tilde{x}^i = x^i + L^i(\eta, x^j)$, explain why the perturbation to the metric, $\delta g_{\mu\nu}$, transforms to

$$\delta\tilde{g}_{\mu\nu} \;=\; \delta g_{\mu\nu} + \left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} \;-\; \delta^\alpha_\mu\,\delta^\beta_\nu\right)\bar{g}_{\alpha\beta} - T\dot{\bar{g}}_{\mu\nu}$$

at linear order, where $\bar{g}_{\mu\nu} = a^2\mathrm{diag}(1,-1,-1,-1)$ is the background metric and overdots denote differentiation with respect to $x^0 \equiv \eta$.

(ii) Noting that

$$\begin{pmatrix} \partial\eta/\partial\tilde{\eta} & \partial\eta/\partial\tilde{x}^i \\ \partial x^i/\partial\tilde{\eta} & \partial x^i/\partial\tilde{x}^j \end{pmatrix} = \begin{pmatrix} 1-\dot{T} & -\partial_i T \\ -\dot{L}^i & \delta^i_j - \partial_j L^i \end{pmatrix} ,$$

show that the metric perturbation variables transform as

$$\begin{aligned} \tilde{\psi} &= \psi - \dot{T} - \mathcal{H}T \\ \tilde{\phi} &= \phi + \mathcal{H}T + \frac{1}{3}\,\partial_i L^i \\ \tilde{B}_i &= B_i + \partial_i T - \dot{L}_i \\ \tilde{E}_{ij} &= E_{ij} - \partial_{\langle i}L_{j\rangle} \,, \end{aligned}$$

where $\mathcal{H} \equiv \dot{a}/a$, spatial indices are raised and lowered with δ_{ij} and angled brackets around indices denote the symmetric trace-free part.

(iii) The peculiar velocity of matter, v^i, transforms as $\tilde{v}^i = v^i + \dot{L}^i$. Give a physical interpretation of this result.

(iv) For scalar perturbations ($E_{ij} = \partial_{\langle i}\partial_{j\rangle}E$, $B_i = \partial_i B$, $v_i = \partial_i v$ etc.), the intrinsic curvature of constant-time hypersurfaces can be shown to be

$$a^2\,{}^{(3)}R \;=\; 4\nabla^2\left(\phi + \frac{1}{3}\,\nabla^2 E\right) .$$

Use this to show that the intrinsic curvature of comoving hypersurfaces – those orthogonal to the matter worldlines – can be written as $a^2\,{}^{(3)}R_{\mathrm{co}} = -4\nabla^2\mathcal{R}$, where

$$\mathcal{R} \;=\; -\phi - \frac{1}{3}\,\nabla^2 E + \mathcal{H}(B+v) \,.$$

Verify the gauge-invariance of this expression for $\mathcal{R}$.

(v) In the conformal Newtonian gauge ($B = E = 0$), the perturbed Einstein equations for the case of adiabatic pressure perturbations and vanishing anisotropic stress are

$$\begin{aligned} \dot{\phi} + \mathcal{H}\phi &= -4\pi G a^2(\bar{\rho} + \bar{P})v \\ \nabla^2\phi &= 4\pi G a^2[\delta\rho - 3\mathcal{H}(\bar{\rho} + \bar{P})v] \\ \ddot{\phi} + 3\mathcal{H}\dot{\phi} + (2\dot{\mathcal{H}} + \mathcal{H}^2)\phi &= 4\pi G a^2(\dot{\bar{P}}/\dot{\bar{\rho}})\delta\rho\,, \end{aligned}$$

and $\phi = \psi$, where $\delta\rho$ is the energy density perturbation and $\bar{\rho}$ and $\bar{P}$ are the background energy density and pressure respectively. By expressing $\mathcal{R}$ in terms of ϕ in the Newtonian gauge, or otherwise, show that

$$-4\,\pi G a^2(\bar{\rho} + \bar{P})\dot{\mathcal{R}} = \frac{\mathcal{H}\dot{\bar{P}}}{\dot{\bar{\rho}}}\,\nabla^2\phi\,.$$

[You may wish to use the Friedmann equation $\mathcal{H}^2 - \dot{\mathcal{H}} = 4\,\pi G a^2(\bar{\rho} + \bar{P})$.] Noting that $\mathcal{R} \sim \phi$ on super-Hubble scales, establish that $\mathcal{R}$ is constant in time on such scales.

(vi) Briefly comment on the significance of the constancy of $\mathcal{R}$ for relating fluctuations produced during inflation to cosmological observables.

4 Outline the quantum generation of almost scale-invariant fluctuations $\delta\Phi$ in a light scalar field during inflation with power spectrum

$$\mathcal{P}_{\delta\Phi}(k) = \left(\frac{H_k}{2\pi}\right)^2 ,$$

where H_k is the Hubble parameter when the mode of wavenumber k exits the Hubble radius.

Tensor perturbations of the metric take the form

$$ds^2 = a^2(\eta)\left[d\eta^2 - (\delta_{ij} + 2E^T_{ij})\, dx^i dx^j\right] ,$$

where E^T_{ij} is symmetric, trace-free and $\delta^{ik}\partial_k E^T_{ij} = 0$. The Einstein-Hilbert action to second-order in E^T_{ij} is

$$S^{(2)} = \frac{M_{\mathrm{Pl}}^2}{2}\int d\eta\, d^3\mathbf{x}\, a^2\left[\dot{E}^T_{ij}\dot{E}^{T\,ij} - \partial_i E^T_{jk}\,\delta^{il}\partial_l E^{T\,jk}\right] ,$$

where M_{Pl} is the reduced Planck mass, overdots denote differentiation with respect to conformal time η, and spatial indices are raised and lowered with δ_{ij}.

(i) Derive the classical equation of motion for E^T_{ij} and show that on super-Hubble scales it has solutions with E^T_{ij} constant in time.

(ii) By expanding E^T_{ij} in Fourier modes as

$$E^T_{ij}(\eta, \mathbf{x}) = \sum_{p=\pm 2}\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, M^{(p)}_{ij}(\mathbf{k})\psi_{(p)}(\eta, \mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}} ,$$

where the symmetric, trace-free polarization tensors $M^{(p)}_{ij}(\mathbf{k})$ are perpendicular to $\mathbf{k}$ and satisfy $M^{(p)}_{ij}(\mathbf{k})[M^{(q)\,ij}(\mathbf{k})]^* = \delta^{pq}$ and $[M^{(p)}_{ij}(\mathbf{k})]^* = M^{(p)}_{ij}(-\mathbf{k})$, show that the action can be written as

$$S^{(2)} = \frac{M_{\mathrm{Pl}}^2}{2}\sum_{p=\pm 2}\int d\eta d^3\mathbf{k}\, a^2\left[\dot{\psi}_{(p)}(\mathbf{k})\dot{\psi}^*_{(p)}(\mathbf{k}) - k^2\psi_{(p)}(\mathbf{k})\psi^*_{(p)}(\mathbf{k})\right] .$$

(iii) By comparing to the action for a massless real scalar field,

$$S^{(2)} = \frac{1}{2}\int d\eta\, d^3\mathbf{x}\, a^2\left\{[\partial_\eta(\delta\Phi)]^2 - (\nabla\delta\Phi)^2\right\} ,$$

argue that tensor perturbations of the metric behave like two independent massless scalar fields with $M_{\mathrm{Pl}}\psi_{(p)}(\eta, \mathbf{k}) \to \delta\Phi(\eta, \mathbf{k})$ for each polarization and hence that inflation generates tensor perturbations with

$$\langle\psi_{(p)}(\mathbf{k})\psi^*_{(q)}(\mathbf{k}')\rangle = \frac{2\pi^2}{k^3}\,\delta_{pq}\mathcal{P}_\psi(k)\delta(\mathbf{k} - \mathbf{k}')$$

on super-Hubble scales, where $\mathcal{P}_\psi(k) = \mathcal{P}_{\delta\Phi}(k)/M_{\mathrm{Pl}}^2$.

(iv) For inflation driven by a slowly-rolling scalar field with potential $V(\bar{\Phi})$, use the slow-roll approximation

$$H^2 \approx \frac{1}{3M_{\rm Pl}^2} V(\bar{\Phi}) \,, \qquad 3H\partial_t\bar{\Phi} \approx -V'(\bar{\Phi}) \,,$$

to show that the tensor spectral index, defined by

$$n_t \equiv \frac{d\ln \mathcal{P}_\psi(k)}{d\ln k} \,,$$

is approximately equal to $-2\,\epsilon_V$, where the slow-roll parameter $\epsilon_V \equiv \frac{1}{2}\, M_{\rm Pl}^2 (V'/V)^2$.

END OF PAPER