UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 27 May, 2010 1:30 pm to 4:30 pm

PAPER 52

GENERAL RELATIVITY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

The signature is $(-+++)$ *and the Riemann curvature tensor convention is*

$$R^i{}_{jkl} = \Gamma^i{}_{jl,k} - \Gamma^i{}_{jk,l} + \Gamma^i{}_{mk}\Gamma^m{}_{jl} - \Gamma^i{}_{ml}\Gamma^m{}_{jk}\,.$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Nordstrøm's theory (1913) is an alternative theory of gravity in which spacetime is endowed with a flat background metric η_{ij} and a scalar field ϕ that is generated via

$$\eta^{ij}\phi_{,ij} = -4\,\pi\phi\,\eta^{ij}T_{ij}\,,$$

where T_{ij} is the energy-momentum tensor of the matter. The *physical* metric g_{ij} that determines the geodesic motion of test particles is then given by

$$g_{ij} = \phi^2\eta_{ij}\,.$$

(a) Assume that $\phi \approx 1$ and that time variations of ϕ are small compared to spatial variations. Let spacetime contain a pressureless perfect fluid. Show that ϕ obeys the same Poisson equation as the gravitational potential in Newtonian theory. Show also that the trajectories $x^\alpha(t)$ of freely falling particles in Newtonian theory coincide with the geodesics in Nordstrøm's theory for non-relativistic velocities $||\dot{x}^\alpha(t)|| \ll 1$.

(b) Is Nordstrøm's theory in agreement with the Pound-Rebka experiment?

(c) Is Nordstrøm's theory consistent with the observed bending of light by the Sun?

Give reasons for your answers.

2

Consider linearized perturbations of Minkowski spacetime with metric

$$g_{ij} = \eta_{ij} + \epsilon\, h_{ij} \,,$$

where $\epsilon \ll 1$. Explain the concept of *gauge* in this context. Show that for a particular choice of gauge (to be specified) the linearized vacuum Einstein equations take the form

$$\Box \bar{h}_{ij} = 0 \,, \tag{1}$$

where $\bar{h}_{ij} = h_{ij} - \frac{1}{2} h_k{}^k \eta_{ij}$ is the trace-reversed perturbation and $\Box = \partial_k \partial^k$ is the flat-space wave operator.

Argue that (1) admits plane-wave solutions of the form

$$\bar{h}_{xx} = -\bar{h}_{yy} = h_+ \,, \qquad \bar{h}_{xy} = h_\times \,, \qquad \bar{h}_{ij} = 0 \text{ otherwise} \,, \tag{2}$$

where h_+ and $h_\times$ are functions of $t - z$ only.

A prototype gravitational wave detector consists of a rigid non-rotating stick, on which two beads can slide freely. Suppose the stick is oriented along the x-axis and a gravitational wave of the form (2) is incident in the z-direction. Find the (proper) separation of the two beads as a function of time if they are initially at rest relative to each other. What happens if the stick is oriented along the z-axis?

3

Schwarzschild spacetime is given by the line element

$$ds^2 = -f\mathrm{d}t^2 + f^{-1}\mathrm{d}r^2 + r^2 d\Sigma^2\,, \tag{1}$$

where $f \equiv 1 - 2M/r$, M is a constant and $d\Sigma^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ denotes the line element on the unit 2-sphere.

Consider a freely falling observer who is moving radially inwards. Show that

$$f\dot{t} = E\,, \qquad f + \dot{r}^2 = E^2\,,$$

where an overdot denotes differentiation with respect to the observer's proper time τ, and E is the observer's energy per unit mass. Specialise to observers starting at rest at infinity so that $E = 1$. Compute the time interval that such observers measure in their rest frame from when they cross the horizon until they arrive at the centre of the black hole.

Show that the observer's four-velocity $U^a = \mathrm{d}x^a/\mathrm{d}\tau$ is given by

$$U = f^{-1}\frac{\partial}{\partial t} - \sqrt{1-f}\,\frac{\partial}{\partial r}\,.$$

Obtain proper time τ from $U_a = -\partial_a \tau$, finding

$$\mathrm{d}\tau = \mathrm{d}t + f^{-1}\sqrt{1-f}\,\mathrm{d}r\,.$$

Now replace Schwarzschild time t with the proper time τ defined by these observers to obtain the Schwarzschild metric in *Painlevé-Gullstrand coordinates*,

$$ds^2 = -\mathrm{d}\tau^2 + \left(\mathrm{d}r + \sqrt{\frac{2M}{r}}\,\mathrm{d}\tau\right)^2 + r^2 d\Sigma^2\,.$$

Explain how this removes the coordinate singularity of the original Schwarzschild metric (1). How does this approach differ from the one based on ingoing Eddington-Finkelstein coordinates discussed in the lectures? Draw the Kruskal diagram of (maximally extended) Schwarzschild spacetime and identify the various regions. Which part of the spacetime is covered by the Painlevé-Gullstrand coordinates?

4

The line element of Minkowski spacetime in spherical polar coordinates is given by

$$ds^2 \;=\; -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 d\Sigma^2 \,,$$

where $d\Sigma^2 \;=\; \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ denotes the line element on the unit 2-sphere. Introduce null coordinates $u = t - r$ and $v = t + r$, then compactified coordinates $p = \tan^{-1} u$ and $q = \tan^{-1} v$, and finally $T = q + p$ and $R = q - p$. Show that the line element can be written in the form

$$ds^2 \;=\; \Omega^{-2} d\hat{s}^2 \,,$$

where $\Omega = 2\cos p \cos q$ is a conformal factor and

$$d\hat{s}^2 \;=\; -\mathrm{d}T^2 + \mathrm{d}R^2 + \sin^2 R \, d\Sigma^2$$

is a conformally related spacetime that should be identified. State the ranges of all the coordinates t, r, θ, ϕ, u, v, p, q, T and R.

Discuss the asymptotic behaviour of geodesics in Minkowski spacetime as seen in the conformally related spacetime. Draw the Penrose diagram, clearly labelling the different parts of its boundary.

Does this spacetime admit particle horizons, future and past event horizons of timelike observers, and Cauchy horizons? Justify your answers.

END OF PAPER