UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

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Monday, 7 June, 2010 1:30 pm to 3:30 pm

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# PAPER 51

# QUANTUM INFORMATION, ENTANGLEMENT AND NONLOCALITY

*You may attempt all* ***FOUR*** *questions.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

(a) Explain briefly what is meant by *quantum teleportation* of an unknown qubit, stating clearly what classical and quantum resources are used.

(b) Describe a protocol for teleporting an unknown state in an $n$ dimensional Hilbert space, and show that your protocol does indeed faithfully implement quantum teleportation.

(c) A, B and C are separated. A and B share a single entangled singlet state of two qubits; B and C also share a single entangled singlet state of two qubits. Is there a protocol, using only local operations and classical communication, which is guaranteed to create an entangled singlet state between A and C? Justify your answer carefully.

**2**

(a) The Hilbert spaces for separated systems $A$ and $B$ have orthonormal bases $\{|i\rangle_A\}_{i=1}^2$ and $\{|i\rangle_B\}_{i=1}^2$ respectively. The bipartite states $|\phi_1\rangle_{AB}$ and $|\phi_2\rangle_{AB}$ are defined by

$$|\phi_1\rangle_{AB} = \frac{1}{\sqrt{2}}|1\rangle_A|1\rangle_B + \frac{1}{\sqrt{2}}|2\rangle_A|2\rangle_B$$

and

$$|\phi_2\rangle_{AB} = \frac{1}{2\sqrt{3}}(|1\rangle_A + |2\rangle_A)(|1\rangle_B + |2\rangle_B) + \frac{\sqrt{2}}{2\sqrt{3}}(|1\rangle_A - |2\rangle_A)(|1\rangle_B - |2\rangle_B)\,.$$

Is it possible to find local unitary operations $U_A$ and $U_B$ acting on the respective Hilbert spaces such that $U_A \otimes U_B|\phi_1\rangle = |\phi_2\rangle$? Justify your answer carefully. (You may cite any relevant theorems without proof, provided you state them carefully.)

(b) What is the *density matrix* corresponding to the probabilistic ensemble of quantum states $\{|\psi_i\rangle\}_{i=1}^n$ with corresponding probabilities $\{p_i\}_{i=1}^n$?

(c) If two ensembles have distinct density matrices, is it possible to find a projective measurement which gives an experimenter some (perhaps probabilistic) information about which ensemble a state in his possession belongs to? Justify your answer carefully.

(d) If two ensembles have the same density matrix, is it possible to find any (not necessarily projective) measurement which gives an experimenter some (perhaps probabilistic) information about which ensemble a state in his possession belongs to? Justify your answer carefully.

**3**

Alice and Bob share a known entangled state $|\psi\rangle = \cos\alpha|00\rangle + \sin\alpha|11\rangle$, with $0 < \alpha < \pi/4$. They wish to convert this state into the state $|\phi\rangle = \cos\beta|00\rangle + \sin\beta|11\rangle$, $\alpha < \beta < \pi/4$, using only local operations and classical communication.

(a) Show that $|\psi\rangle$ cannot be reliably converted into $|\phi\rangle$. (You may cite the *majorization condition* without proof.)

(b) Give an upper bound on the maximal probability of a successful transformation. (You may cite any relevant theorem(s) without proof).

(c) Develop a strategy (by making a suitable adaptation of the Procrustean Method or otherwise) that allows Alice and Bob to achieve their goal with the maximal probability of success. You should explicitly demonstrate that your result agrees with (b).

Assume that Alice and Bob share $n$ identical copies of the state $|\psi\rangle$, where $n$ is very large, and want to transform them into $m(n)$ copies of $|\phi\rangle$ using local operations and classical communication only.

(d) Determine the maximal possible value of the asymptotic rate, $\lim_{n\to\infty} \dfrac{m(n)}{n}$.

(e) Compare your result from the previous section with the number of states $|\phi\rangle$, which would be obtained if Alice and Bob acted on each of the $n$ states $|\psi\rangle$ separately using the protocol developed in section (c).

**4**

(i) A *purification* of a mixed quantum state $\rho_A$ of system $A$ is a pure state $|\psi\rangle_{AB}$ of systems $A$ and $B$ such that

$$tr_B(|\psi\rangle_{AB}\langle\psi|_{AB}) = \rho_A$$

Find two purifications of the state

$$\rho_A = \tfrac{1}{2}|0\rangle_A\langle 0|_A + \tfrac{1}{2}|1\rangle_A\langle 1|_A + \tfrac{1}{6}|1\rangle_A\langle 0|_A + \tfrac{1}{6}|0\rangle_A\langle 1|_A\,.$$

(ii) The four Bell states are:

$$|\phi^{\pm}\rangle_{AB} = (|00\rangle_{AB} \pm |11\rangle_{AB})/\sqrt{2}$$

$$|\psi^{\pm}\rangle_{AB} = (|01\rangle_{AB} \pm |10\rangle_{AB})/\sqrt{2}$$

Alice and Bob are given two identical copies of one of the Bell states. Provide a protocol for them to distinguish which of the four Bell states they have, using only local operations and classical communication on their two copies.

(iii) Alice Bob and Charlie share a GHZ state,

$$|GHZ\rangle_{ABC} = (|000\rangle_{ABC} + |111\rangle_{ABC})/\sqrt{2}\ .$$

Give a protocol for converting this into the Bell state $|\psi^{+}\rangle_{AB}$ between Alice and Bob, using only local operations and classical communication between any of the three parties.

## END OF PAPER