UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 7 June, 2010 9:00 am to 11:00 am

PAPER 50

CONTROL OF QUANTUM SYSTEMS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Open-loop Control Design

(a) Let $\mathcal{H}$ be a finite-dimensional Hilbert space, and let $\rho(t)$ and $\rho_d(t)$ be density operators on $\mathcal{H}$, representing the system and target state, respectively, satisfying

$$\dot{\rho}(t) \;=\; -\,i\,[H_0 + f(t)H_1,\, \rho(t)]\;, \qquad \dot{\rho}_d(t) \;=\; -\,i\,[H_0,\, \rho_d(t)]\;,$$

where H_0 and H_1 are Hermitian operators on $\mathcal{H}$ and $f(t)$ is a real-valued function. Let

$$V(\rho(t),\, \rho_d(t)) \;=\; \frac{1}{2}\,\|\rho(t) - \rho_d(t)\|^2\,,$$

where $\|x\| = \sqrt{\mathrm{Tr}(x^\dagger x)}$ is the Hilbert-Schmidt norm.

Show that setting $f(t) = \mathrm{Tr}(\rho_d(t)[-iH_1,\, \rho(t)])$ implies $\dot{V}(t) \leqslant 0$, and explain why this shows that the distance of $\rho(t)$ from $\rho_d(t)$ is monotonically decreasing.

Hint: $\mathrm{Tr}([-iH_0,\, \rho_d(t)]\rho(t)) \;=\; -\,\mathrm{Tr}(\rho_d(t)[-iH_0,\, \rho(t)])$.

(b) Is the argument in part (a) sufficient to conclude that the trajectory $\rho(t)$ of every initial state $\rho(0)$ will converge to the target trajectory $\rho_d(t)$, i.e., $\|\rho(t) - \rho_d(t)\| \to 0$ for $t \to \infty$? Briefly justify why or why not. Is it sufficient if we assume $\rho(0)$ and $\rho_d(0)$ have the same spectrum?

(c) The scheme in part (a) essentially provides a feedback control law that steers the system from some initial state $\rho(0)$ to a desired target state $\rho_d(t)$. Could this feedback law be used for measurement-based feedback control for quantum systems? If not, why not?

(d) Explain how we can formulate different quantum control tasks as optimal control problems, and discuss how these can be solved numerically. Explain the equations that need to be solved, parametrization of thecontrols, etc.

(e) Outline how optimal control pulses can be implemented in the laboratory using spectral pulse shaping techniques. Sketch the core components of a typical pulse shaper and explain what functions they perform.

2 Dissipative Dynamics and Steady States

Let $\mathcal{H}$ be an N-dimensional Hilbert space. Define the Hilbert-Schmidt (HS) inner product for operators A, B on $\mathcal{H}$ by $\langle A|B\rangle = \mathrm{Tr}(A^\dagger B)$ and let $\{\sigma_k\}_{k=1}^{N^2}$ be an orthonormal basis w.r.t. the HS inner product for the Hermitian matrices on $\mathcal{H}$, with $\sigma_{N^2} = \frac{1}{\sqrt{N}}\,\mathbb{I}$, where $\mathbb{I}$ is the identity matrix.

(a) Show that any operator A on $\mathcal{H}$ can be written as $A = \sum_{k=1}^{N^2} a_k\sigma_k$ with $a_k = \mathrm{Tr}(\sigma_k A)$, and show that the coordinate vector $\vec{a} = (a_1, \ldots, a_{N^2})$ is real if A is Hermitian.

(b) Assume ρ satisfies the quantum Liouville equation ($\hbar = 1$)

$$\dot\rho(t) = -i\big[H,\rho(t)\big] + \sum_d \mathbb{D}\,[V_d]\rho(t)\,, \qquad \mathbb{D}\,[V]\rho = V\rho V^\dagger - \frac{1}{2}\Big(V^\dagger V\rho + \rho V^\dagger V\Big)\,.$$

Show that the coordinate vector $\vec{r}$ of ρ satisfies the matrix differential equation $\dot{\vec{r}}(t) = (\mathbf{L} + \sum_d \mathbf{D}^{(d)})\vec{r}(t)$ where $\mathbf{L}$ and $\mathbf{D}^{(d)}$ are $N^2\times N^2$ matrices with

$$L_{mn} = \mathrm{Tr}\Big(iH[\sigma_m,\sigma_n]\Big)\,, \qquad D^{(d)}_{mn} = \mathrm{Tr}\Big(V_d^\dagger\sigma_m V_d\sigma_n\Big) - \frac{1}{2}\,\mathrm{Tr}\,\Big(V_d^\dagger V_d\{\sigma_m,\sigma_n\}\Big)$$

where $[A,B] = AB - BA$ is the usual matrix commutator and $\{A,B\} = AB + BA$ the anticommutator. Furthermore show that $\dot r_{N^2} = 0$, define the reduced Bloch vector $\vec{s}$ in terms of $\vec{r}$, and show that it satisfies an affine linear equation $\dot{\vec{s}} = \mathbf{A}\,\vec{s} + \vec{c}$.

(c) Define the notion of a steady state and characterize the steady states of the Bloch equation $\dot{\vec{s}}(t) = \mathbf{A}\,\vec{s}(t) + \vec{c}$ as a linear equation in terms of the rank of $\mathbf{A}$. Does the Bloch equation always have a steady state? When does the Bloch equation have a unique steady state?

(d) Give a necessary and sufficient condition for attractivity of a steady state in terms of the eigenvalues of $\mathbf{A}$.

(e) Show that for $(\alpha,\beta)\neq(0,0)$, the state $(\alpha\beta, 0, -\beta^3)^T/D$ with $D = \alpha^2+\beta^2(\beta^2+1)$ is the unique steady state of the single-qubit Bloch equation

$$\frac{d}{dt}\begin{pmatrix} x(t)\\ y(t)\\ z(t)\end{pmatrix} = \begin{pmatrix} -\beta^2 & 0 & -\alpha\\ 0 & -1 & 0\\ \alpha & 0 & -\beta^2-1\end{pmatrix}\begin{pmatrix} x(t)\\ y(t)\\ z(t)\end{pmatrix} - \begin{pmatrix} 0\\ 0\\ \beta\end{pmatrix}.$$

Is this state attractive? Hint: Consider the eigenvalues of $\mathbf{A}$.

3 Controllability and Spin Chains

(a) Briefly explain the general concepts of reachability and controllability in control theory, and show how the generic concept of controllability can be applied to quantum systems to derive the notions of unitary operator controllability, density operator controllability and pure-state controllability.

(b) Consider a control-linear Hamiltonian system with

$$H[f_m(t)] \;=\; H_0 + \sum_{m=1}^{M} f_m(t)\, H_m \,,$$

where $f_m(t)$ are control functions (usually in $L^2(0,T)$) and $H_m,\;\; m = 0, \dots, M$ are Hermitian operators on a Hilbert space $\mathcal{H}$. Assuming $\dim \mathcal{H} = N$, give necessary and sufficient conditions for each of the controllability notions above in terms of the dynamical Lie algebra of the system.

Next, let us apply some of these concepts to spin chains. A standard model for the system Hamiltonian of a network of spin-$\frac{1}{2}$ particles is the Heisenberg model,

$$H_S \;=\; \sum_{1 \leqslant m < n \leqslant N} \alpha_{m\,n}\, X_m\, X_n + \beta_{m\,n}\, Y_m\, Y_n + \gamma_{m\,n}\, Z_m\, Z_n \,,$$

where the operator X_n (respectively, Y_n, Z_n) denotes an N-fold tensor product, all of whose factors are the identity I, except for the nth factor, which is the Pauli matrix X (respectively, Y, Z). E.g. for a network with 3 spins we would have $X_2 = I \otimes X \otimes I$. The Pauli matrices are given as usual by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) Let $S = \sum_{n=1}^{N} Z_n$. Using the identities $XY = iZ$, $YZ = iX$, $ZX = iY$ show that $[X_m\, X_n + Y_m\, Y_n,\; Z_m + Z_n] = 0$, $[Z_m\, Z_n,\; Z_m + Z_n] = 0$, and that the Hamiltonian H_S above commutes with S if $\alpha_{mn} = \beta_{mn}$ for all (m,n).

(d) Assume we have control over a single spin in the network with a control Hamiltonian $H_1 = Z_k$ for some fixed $k \in \{1, \dots, N\}$. We clearly have $[H_1, S] = 0$. Together with the result that $[H_S, S] = 0$, what conclusions can you draw from this about the control system with $H = H_S + f(t)\, H_1$. Is the system controllable? If not, why not?

(e) Write down the matrix representation of S for $N = 3$ and hence show that the symmetry operator S is diagonal and has $N+1$ distinct eigenvalues of the form $N,\, N-2,\, N-4,\, \dots,\, -N+2,\, -N$.

(f) Consider the single-excitation subspace of a linear chain with $XX + YY$ coupling, i.e., $\alpha_{n,n+1} = \alpha_{n+1,n} = c_n$, $\beta_{n,n+1} = \beta_{n+1,n} = c_n$ and $\alpha_{mn} = \beta_{mn} = 0$ otherwise $\gamma_{mn} = 0$. The Hamiltonian $H_S^{(1)}$ on the relevant subspace takes the form $H_S^{(1)} = A$, where A is an $N \times N$ matrix with $A_{n,\,n+1} = A_{n+1,\,n} = c_n$ and zeros otherwise, and the control Hamiltonian $H_1 = Z_1$ takes the form $B = \text{diag}(-1, 1, \ldots, 1)$. For $N = 3$, we can generate the following linearly independent matrices (I_N identity matrix):

$$\begin{aligned} e_{11} &= i\,(I_N - B)/2\,; & y_{12} &= [e_{11},\, [iA, e_{11}]]/c_1\,; & x_{12} &= [y_{12},\, e_{11}]\,;\\ e_{22} &= e_{11} - [x_{12},\, y_{12}]\,; & y_{23} &= (iA - c_1\, y_{12})/c_2\,; & x_{23} &= [y_{23},\, e_{22}]\,;\\ e_{33} &= e_{22} - [x_{23},\, y_{23}]\,; & y_{13} &= [y_{12},\, y_{23}]\,, & x_{13} &= [x_{12},\, x_{23}]\,. \end{aligned}$$

You do *not* need to calculate the commutators or verify linear independence.

What conclusions can you draw from this about the Lie algebra, and the controllability, of the system on the single-excitation subspace? If you were told in addition that the dimension of the Lie algebra on the entire Hilbert space was 9, what would this tell you about the Lie algebra generated by $H = H_0 + f(t)\, H_1$?

(g) Consider a homogeneous linear chain as above but with a control that modulates the coupling between the first two spins. For $N = 3$ the system and control Hamiltonian on the single excitation subspace take the explicit form

$$H_S^{(1)} = A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad H_c^{(1)} = C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Show that $x^T J + Jx = 0$ for $x = iA$ and $x = iC$ and $J = \text{diag}(1, -1, 1)$. What conclusions do you draw from this about symmetries, the Lie algebra and controllability of the system $A + f(t)C$?

END OF PAPER