UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2010 9:00 am to 12:00 pm

PAPER 5

METHODS IN ANALYSIS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Fix $n \in \mathbb{N}$, and $1 < p < \infty$. Show that the dual of $L^p(\mathbb{R}^n)$ is the space $L^q(\mathbb{R}^n)$, where $1 < q < \infty$ is such that $1/p + 1/q = 1$ (in other words, show that there is an isometric isomorphism between $(L^p(\mathbb{R}^n))^*$ and $L^q(\mathbb{R}^n)$).

[*You may use the following fact without proving it.*
If K is a closed linear subspace of $L^p(\mathbb{R}^n)$, and if $f \in L^p(\mathbb{R}^n)$ is not in K, then there exists $h \in K$ with $\inf_{g\in K} \|f - g\|_{L^p} = \|f - h\|_{L^p}$ and such that

$$\int |f - h|^{p-2}\,(\overline{f - h})\, g \,=\, 0$$

for all $g \in K$.]

(b) State and prove the Banach-Alaoglu Theorem for $L^p(\mathbb{R}^n)$, for $1 < p < \infty$.

[*You may use the fact that $L^p(\mathbb{R}^n)$ is separable for all $1 < p < \infty$ without proving it.*]

(c) Show with an example that the Banach-Alaoglu Theorem fails if $p = 1$.

2

(a) State and prove the Hardy-Littlewood-Sobolev inequality.

(b) For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ and $q > 1$, define

$$\langle f\rangle_{w,q} \,=\, \sup_{\alpha>0}\, \alpha\, \left|\{x \in \mathbb{R}^n : |f(x)| > \alpha\}\right|^{1/q}$$

$$\|f\|_{w,q} \,=\, \sup_{A}\, |A|^{-1/q'} \int_A |f(x)|\, dx\,.$$

In the definition of $\|f\|_{w,q}$, the supremum is taken over all measurable sets A with $|A| < \infty$, and $1/q + 1/q' = 1$. Show that there exist $C_{1,q}, C_{2,q} > 0$ such that

$$C_{1,q}\langle f\rangle_{w,q} \,\leqslant\, \|f\|_{w,q} \,\leqslant\, C_{2,q}\langle f\rangle_{w,q}\,.$$

for all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$.

3

(a) Assume that $n \geqslant 3$. Suppose that $f, f_j \in H^1(\mathbb{R}^n)$ for every $j \in \mathbb{N}$ are such that $f_j \to f$ weakly in $H^1(\mathbb{R}^n)$ as $j \to \infty$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded set and that χ_Ω denotes the characteristic function of Ω. Show that $\chi_\Omega f_j \to \chi_\Omega f$ strongly in $L^q(\mathbb{R}^n)$, for all $q < 2n/(n-2)$.

(b) State and prove the Poincaré inequality for $f \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded, connected, open set having the cone-property, and where $p < n$.

[*In the proof you can make use of the Rellich-Kondrashov Theorem on general sets having the cone-property.*]

4

(a) Prove that the maps defined by

$$\mathrm{PV}\frac{1}{x}(\phi) = \lim_{\varepsilon \to 0} \left[\int_{-\infty}^{-\varepsilon} dx\, \frac{\phi(x)}{x} + \int_{\varepsilon}^{\infty} dx\, \frac{\phi(x)}{x} \right]$$

$$\frac{1}{x + i0}\,(\phi) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} dx\, \frac{1}{x + i\varepsilon} \phi(x)$$

for all $\phi \in \mathcal{D}(\mathbb{R})$ are distributions (PV stands for "Principal Value"). Show moreover that

$$\frac{1}{x + i0} = \mathrm{PV}\,\frac{1}{x} + i\pi\delta$$

where $\delta(\phi) = \phi(0)$ for all $\phi \in \mathcal{D}(\mathbb{R})$.

(b) Let f_j be a sequence in $H^1(\mathbb{R}^n)$. Assume that there exist $g, h_1, \ldots, h_n \in L^2(\mathbb{R}^n)$ such that $f_j \to g$ weakly in $L^2(\mathbb{R}^n)$ and $\partial_\ell f_j \to h_\ell$ weakly in $L^2(\mathbb{R}^n)$ for every $\ell = 1, \ldots, n$, as $j \to \infty$. Prove that $g \in H^1(\mathbb{R}^n)$ and that $\partial_\ell g = h_\ell$ for every $\ell = 1, \ldots, n$.

5

(a) Suppose that $\Omega \subset \mathbb{R}^n$ is open and let $f \in L^1_{\text{loc}}(\Omega)$ be real valued. What does it mean for f to be *subharmonic* on Ω? What does it mean for f to be *superharmonic* on Ω? What does it mean for f to be *harmonic* on Ω?

(b) State and prove the strong maximum principle for subharmonic functions.

(c) State and prove Harnack's inequality.

(d) Suppose that u_j is a sequence of harmonic functions on Ω with $u_j \to u$ pointwise and such that u_j is uniformly bounded on any compact subset $K \subset \Omega$. Prove that $u_j \to u$ uniformly on every compact subset $K \subset \Omega$ and that u is harmonic (you should assume here that the function $u_j = \widetilde{u}_j$ is given by the representative of the equivalence class of u_j which is continuous and harmonic at every point $x \in \Omega$).

END OF PAPER