UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2010 1:30 pm to 4:30 pm

PAPER 48

QUANTUM INFORMATION THEORY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let $\rho = \sum_{i=1}^{k} p_i\rho_i$, where p_i are probabilities satisfying $\sum_{i=1}^{k} p_i = 1$, and ρ_i are states which have supports on mutually orthogonal subspaces. Prove that

$$S\left(\sum_i p_i\rho_i\right) = H(\mathbf{p}) + \sum_i p_i S(\rho_i)$$

where $S(\rho)$ denotes the von Neumann entropy of ρ, $\mathbf{p} = \{p_i\}_{i=1}^{k}$ and $H(\mathbf{p})$ denotes the corresponding Shannon entropy.

(b) State the subadditivity and strong subadditivity properties of the von Neumann entropy of a state.

(c) Using the result of part (a), and the subadditivity of the von Neumann entropy, prove that

$$S\left(\sum_{i=1}^{r} p_i\rho_i\right) \geqslant \sum_{i=1}^{r} p_i S\left(\rho_i\right).$$

2

Suppose a quantum system Q in a state ρ is sent through a noisy quantum channel Φ. Let $|\Psi_{RQ}^{\rho}\rangle$ denote the purification of the state ρ, with R being the reference system used for the purification.

(a) Give a diagrammatic representation of the action of the channel on the state $|\Psi_{RQ}^{\rho}\rangle$, and define the coherent information $I_c(\Phi,\rho)$ of the quantum channel Φ with respect to the input state ρ.

(b) Using the fact that every completely positive trace-preserving map can be realized through a unitary evolution of a larger system, prove that $I_c(\Phi,\rho)$ is less than or equal to the von Neumann entropy of ρ.

(c) Let Φ_1 and Φ_2 denote two completely positive trace-preserving (CPTP) maps, whose composition is denoted by $\Phi_2 \circ \Phi_1$. Using the strong subadditivity property of the von Neumann entropy, prove the quantum data processing inequality:

$$I_c(\Phi_1,\rho) \geqslant I_c(\Phi_2 \circ \Phi_1,\, \rho).$$

3

(a) The fidelity of two states ρ and σ is defined as

$$F(\rho, \sigma) := tr\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}\,.$$

(i) Justify that

$$F(\rho, \sigma) = F(\sigma, \rho)\,,$$

clearly stating any other result which you wish to employ.

(ii) Prove that

$$F(\rho \otimes \sigma, \rho' \otimes \sigma') = F(\rho, \rho')\,F(\sigma, \sigma')\,,$$

where ρ, σ, ρ' and σ' denote states of a quantum-mechanical system.

(iii) Prove that if AB is a bipartite system then

$$F(\rho_{AB}, \sigma_{AB}) \leqslant F(\rho_A, \sigma_A)$$

where ρ_A and σ_A denote the reduced states of the subsystem A when AB is in the state ρ_{AB} and σ_{AB} respectively,

(b) Suppose Alice wants to send classical messages to Bob through multiple uses of a memoryless qubit depolarizing channel Φ defined as follows:

$$\Phi(\rho) := p\rho + (1-p)\,\frac{I}{2}\,.$$

Let $\mathcal{I}(\Phi)$ denote the maximum amount of information (in bits) that she could send to Bob per use of the channel, when she is restricted to use only product-state inputs. Prove that

$$\mathcal{I}(\Phi) \leqslant 1 - h(q)\,,$$

for some $0 \leqslant q \leqslant 1$, where $h(q)$ denotes the binary entropy. Determine q.

4

(a) Suppose Alice has a qubit which is in a pure state unknown to her and which she wishes to send to Bob. However, she does not have any quantum channel at her disposal and is only allowed to communicate classically with Bob. Under what condition can she achieve her goal? Give a detailed justification of your answer.

(b) Prove that the Holevo capacity is superadditive, i.e., for two quantum channels Φ_1 and Φ_2:

$$\chi^*(\Phi_1 \otimes \Phi_2) \geqslant \chi^*(\Phi_1) + \chi^*(\Phi_2)\,.$$

Here we have used the notation $\chi^*(\Phi)$ to denote the Holevo capacity of a quantum channel Φ.

5

(a) State the generalized measurement postulate. When does a generalised measurement reduce to a projective measurement?

(b) Write the Schmidt decomposition of a bipartite pure state $|\psi_{AB}\rangle$, and define its Schmidt number.

(c) Find the Schmidt number of the state

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |10\rangle + |01\rangle\right).$$

(d) Find a Schmidt decomposition of the two qubit state

$$|\psi\rangle_{AB} = \frac{1}{2}(1-i)|01\rangle + \frac{1}{2}(1+i)|10\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B\,.$$

END OF PAPER