UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 8 June, 2010 9:00 am to 11:00 am

PAPER 47

SOLITONS AND INSTANTONS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Explain Derrick's theorem on the non-existence of finite energy *static* solutions for a scalar field theory in three or more space dimensions.

Explain how the concept of *non-topological* soliton can lead to the existence of soliton solutions in complex scalar field theories in any space dimension.

For the equation in one space dimension

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \phi = |\phi|^2 \phi \,, \qquad \phi(t,x) \in \mathbb{C}$$

there exist non-topological solitons of the form

$$\phi(t,x) = e^{i\omega t} f(x) \qquad \text{with} \quad \omega^2 < 1 \,,$$

where $f(x) = f(-x) \in \mathbb{R}$ and $\lim_{x \to \pm\infty}(|f(x)| + |f'(x)|) = 0$. It is given explicitly by

$$f(x) = \sqrt{2(1-\omega^2)}\, sech((1-\omega^2)^{1/2}\, x) \,.$$

(*You do not need to show that this is a solution.*) Show, either from the equation or from the formula for f, that

$$(f')^2 = (1-\omega^2)\, f^2 - f^4/2 \,.$$

Write down the Lorentz transformations of these solitons to obtain solutions describing solitons moving along straight lines $x = x_0 + ut$ at arbitrary velocity $u \in (-1, +1)$. Write down expressions (as integrals) for the energy E and momentum P of these solutions at fixed time t, and show that they obey the relativistic energy-momentum relation $E^2 = P^2 + M^2$ with

$$M = \int_{-\infty}^{+\infty} \Big[(f')^2 + \omega^2 f^2 \Big]\, dx \,.$$

[*You do not need to evaluate this integral explicitly.*]

2

Derive the Euler-Lagrange equation of motion for the pure $SU(2)$ Yang-Mills energy on four dimensional Euclidean space $\mathbb{R}^4$

$$V = \int_{\mathbb{R}^4} F^a_{\mu\nu} F^a_{\mu\nu} \, d^4x$$

(Here $F^a_{\mu\nu}$ is the field associated to the gauge potential A^a_μ, and both take values in the Lie algebra $su(2)$ on which a standard orthonormal basis $e_a = -\frac{i}{2}\sigma_a$, where σ_a are the Pauli matrices, is used. Greek letters are for spatial indices $\mu, \nu \ldots \in \{1,2,3,4\}$ and the $su(2)$ indices $a \ldots$ take values in $\{1,2,3\}$; the summation convention is assumed.)

Define what it means for F to be anti-self-dual. State and prove the Bianchi identity, and hence show that if F is anti-self-dual the equation of motion is automatically satisfied.

Show that the anti-self-dual solutions give minimum values of V amongst all gauge potentials with finite energy $V < \infty$ and having a fixed given value for the integral

$$\int_{\mathbb{R}^4} \epsilon_{\mu\nu\kappa\lambda} F^a_{\mu\nu} F^a_{\kappa\lambda} \, d^4x \, . \qquad (1)$$

What is the significance of the integral (1)?

3

Consider the following generalization of the two dimensional static abelian Higgs model, with energy

$$V(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^2} \frac{1}{|\Phi|^2} B^2 + |(\nabla - iA)\Phi|^2 + \frac{|\Phi|^2}{4} (1 - |\Phi|^2)^2 \, d^2x$$

where $\Phi(x) \in \mathbb{C}$ and $A = A_1 \, dx^1 + A_2 \, dx^2$ is the magnetic potential with magnetic field $B = \partial_1 A_2 - \partial_2 A_1$. You may assume that all fields are smooth, and that B, $|(\nabla - iA)\Phi|$, $|1 - |\Phi|^2|$ decay to zero rapidly as $|x| \to \infty$ and

$$\int_{\mathbb{R}^2} B \, d^2x = \lim_{R \to +\infty} \int_{|x| = R} A_j \, dx^j = 2\pi N$$

for some positive integer N.

Write down the second order Euler-Lagrange equations of motion associated to V.

Work out a Bogomolny decomposition for V, and derive a pair of first order equations for A, Φ whose solutions would give minimum energy solutions to the Euler-Lagrange equations.

Derive a second order equation for $u = \ln |\Phi|^2$ and explain how this equation could be used to produce multi-vortex solutions of the first order equations which you previously derived.

4

Explain by means of examples the notion of symmetry for gauge theories, illustrating your answer with particular reference to the radial symmetry of the 't-Hooft-Polyakov monopole solutions:

$$\Phi^a(x) \,=\, f(r)\,\frac{x^a}{r}\,, \qquad A_i^a(x) \,=\, \epsilon_{iaj}\,x^j\,\alpha(r) \qquad r \,=\, |x| \tag{1}$$

for the static $SU(2)$ Yang-Mills-Higgs theory derived from the energy functional:

$$V_\lambda(A,\Phi) \,=\, \int_{\mathbb{R}^3} \Big[B_i^a B_i^a + D_i\,\Phi^a D_i\,\Phi^a + \lambda(1 - \Phi^a\,\Phi^a) \Big]\, d^3x$$

(Here $B_i^a \,=\, \frac{1}{2}\,\epsilon_{ijk}\,F_{jk}^a$ where F_{jk}^a is the field associated to the gauge potential A_j^a, and Φ^a is the Higgs field, and all these fields take values in the Lie algebra $su(2)$ on which a standard orthonormal basis $e_a \,=\, -\,\frac{i}{2}\,\sigma_a$, where σ_a are the Pauli matrices, is used. The spatial and $su(2)$ indices $i,\, j\, \ldots$ and $a\, \ldots$ take values in $\{1,2,3\}$, and the summation convention is assumed. The covariant derivative is $D_k\Phi = \nabla_k\Phi + [A_k,\Phi]$.)

Write down the Euler-Lagrange equations of motion associated to the energy functional V_λ.

The following formulae give the form of the magnetic field B_k^a and the covariant derivative $D_k\Phi^a$ which follow for the radially symmetric monopoles above:

$$B_k^a \,=\, \Big(r\alpha' + 2\alpha\Big)\delta_{ak} - \left(\frac{\alpha'}{r} - \alpha^2\right) x^a x^k \tag{2}$$

and

$$D_k\,\Phi^a \,=\, \left(\frac{f}{r} + r\alpha f\right)\delta_{ak} + \frac{1}{r}\left(\left(\frac{f}{r}\right)' - \alpha f\right) x^a x^k \tag{3}$$

Derive (3) from (1), and show that the Bogomolny equations

$$B_k^a \,=\, -D_k\,\Phi^a\,. \tag{4}$$

reduce to a pair of coupled ordinary differential equations for α and f. (*You do not need to solve these equations.*)

Prove that any solution of (4) obeys the Euler-Lagrange equations of motion for the energy functional $V_{\lambda=0}$.

END OF PAPER