UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 28 May, 2010 9:00 am to 12:00 pm

PAPER 46

ADVANCED QUANTUM FIELD THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The rotation of a rigid body is described by a single angular coordinate θ with the usual periodic identification $\theta \sim \theta + 2\pi$. The evolution of θ in time t is governed by the Lagrangian $L = \Lambda\dot{\theta}^2/2$ where Λ is a constant with the dimensions of length and $\dot{\theta} = d\theta(t)/dt$.

The corresponding quantum system has energy eigenstates $|n\rangle$ labeled by an integer n, with eigenvalue $E_n = n^2\hbar^2/2\Lambda$ which satisfy a completeness relation,

$$\sum_{n=-\infty}^{+\infty} \frac{1}{2\pi} |n\rangle \langle n| \quad = \quad \mathbb{I}$$

where $\mathbb{I}$ denotes the identity operator. The inner product between normalised eigenstates $|\theta\rangle$ of the angular coordinate and energy eigenstates is $\langle\theta|n\rangle = \exp(in\theta)$.

(a) State without proof a path integral formula for the transition amplitude from an initial state with $\theta = \theta_i \in [0, 2\pi]$ at $t = 0$ to a final state $\theta = \theta_f \in [0, 2\pi]$ at time $t = T$. Evaluate this transition amplitude using operator methods leaving the answer in the form of an infinite sum over the energy eigenvalues of the system.

(b) Find all possible classical paths satisfying the boundary conditions described in a) and evaluate the classical action of each path.

(c) Use the identity,

$$\sum_{n=-\infty}^{+\infty} \frac{1}{2\pi} f\left(\frac{n}{2\pi}\right) \exp(in\Theta) \quad = \quad \sum_{m=-\infty}^{+\infty} \tilde{f}\left(\Theta + 2\pi m\right),$$

which you may assume holds for any function f for which the integral,

$$\tilde{f}(k) \quad = \quad \int_{-\infty}^{+\infty} dx\, f(x) \, \exp(2\pi ikx),$$

can be defined by analytic continuation, to write the transmission amplitude as a sum over contributions to the path integral from the classical paths considered in (b).

2

Complex Grassmann variables θ_i and $\bar{\theta}_i$, with $i = 1, 2, \ldots, N$ satisfy

$$\{\theta_i, \theta_j\} = \{\bar{\theta}_i, \bar{\theta}_j\} = \{\theta_i, \bar{\theta}_j\} = 0$$

for all i and j. Throughout this question we adopt a summation convention where repeated indices are summed over.

Define the Grassmann integration measure,

$$\int d^N\bar{\theta}\, d^N\theta$$

and show that it is invariant under the transformation,

$$\begin{aligned} \theta_i &\rightarrow \theta'_i = U_{ij}\theta_j \\ \bar{\theta}_i &\rightarrow \bar{\theta}'_i = U^*_{ij}\bar{\theta}_j \end{aligned}$$

where U is an $N \times N$ unitary matrix (ie $UU^\dagger$ equals the unit matrix). Hence show that,

$$\int d^N\bar{\theta}\, d^N\theta\, \exp\left(-\bar{\theta}_i\, B_{ij}\, \theta_j\right) = \det B$$

for any Hermitian matrix B.

Now consider the following integral where the Grassmann variables introduced above are coupled to N ordinary bosonic variables x_j, $j = 1, 2, \ldots, N$,

$$\mathcal{I}(g) = \int d^N\bar{\theta}\, d^N\theta\, d^N x\, \exp(-S)$$

where,

$$S = \frac{1}{2} x_i\, A_{ij}\, x_j + \bar{\theta}_i\, B_{ij}\, \theta_j - g\, x_k\, \bar{\theta}_i\, C^{(k)}_{ij}\, \theta_j\,,$$

with sums over repeated indicies i, j and k implied. Here A, B and $C^{(k)}$, for $k = 1, \ldots, N$, are $N \times N$ Hermitian matrices and g is a dimensionless coupling constant. In addition the eigenvalues of A are positive. Prove that,

$$\mathcal{I}(g) = \det B \int d^N x\, \exp\left(-\frac{1}{2} x_i\, A_{ij}\, x_j - V_{\text{eff}}(x)\right)$$

where,

$$V_{\text{eff}} = \sum_{m=1}^{\infty} \frac{g^m}{m!}\, x_{k_1}\, x_{k_2} \ldots x_{k_m}\, V^{(m)}_{k_1 k_2 \ldots k_m}$$

with

$$V^{(m)}_{k_1 k_2 \ldots k_m} = (m-1)!\, \text{Tr}\left[B^{-1} C^{(k_1)} B^{-1} C^{(k_2)} \ldots B^{-1} C^{(k_m)}\right]$$

Here $\text{Tr}[X]$ denotes the trace of an $N \times N$ matrix X. *Hint: you may find the identity* $\log(\det X) \equiv \text{Tr}[\log X]$ *useful.*

Hence write down Feynman rules for the expansion of $\mathcal{I}(g)/\mathcal{I}(0)$ in powers of g.

3

Write an essay on the β-function of ϕ^4 scalar field theory in four dimensions. Your account should start from the Callan-Symanzik equation (which need not be derived) and explain how the equation is solved in simple cases. You should include a derivation of the one-loop β-function and of the corresponding one-loop running coupling.

You may use without proof the following relation between the bare and renormalised mass and coupling of the theory, calculated at one-loop in dimensional regularisation (with $d = 4 - \varepsilon$) in the MS renormalisation scheme,

$$\begin{aligned} \lambda_B &= \lambda\mu^{\varepsilon}\left(1 + \frac{3\lambda}{16\pi^2\varepsilon}\right) \\ m_B^2 &= m^2\left(1 + \frac{\lambda}{16\pi^2\varepsilon}\right) \end{aligned}$$

up to corrections of higher order in λ.

4

The gauge-fixed Lagrangian for $SU(N)$ Yang-Mills theory is given as,

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} - \frac{1}{2\zeta}\,\partial^\mu A^a_\mu \partial^\nu A^a_\nu - \partial^\mu \bar{c}^a\,(\mathcal{D}_\mu c)^a$$

where $a = 1, \ldots, N^2 - 1$ with a summation convention for repeated indices and,

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu \\ (\mathcal{D}_\mu c)^a &= \partial_\mu c^a + g f^{abc}\, A^b_\mu c^c \end{aligned}$$

where f^{abc} are structure constants for $SU(N)$.

By defining an appropriate generating functional, calculate the following position space Green's functions at tree level,

i) $$\langle \Omega \left| \mathrm{T}\left\{ A^a_\mu(x)\; A^b_\nu(y) \right\} \right| \Omega \rangle$$

ii) $$\langle \Omega \left| \mathrm{T}\left\{ c^a(x)\; \bar{c}^b(y) \right\} \right| \Omega \rangle$$

where T denotes time-ordering and $|\Omega\rangle$ is the vacuum state. You may use the standard path integral representation of these Green's functions without proof.

Identify the specific term in the Lagrangian giving rise to the ghost-gluon vertex. Explain briefly the form of the corresponding Feynman rule.

END OF PAPER