UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 2 June, 2010 1:30 pm to 4:30 pm

PAPER 45

THE STANDARD MODEL

*Attempt question **ONE** and no more than **TWO** other questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The Weinberg Salam model has a gauge group $SU(2)_T \times U(1)_Y$ with generators (T_i, Y) and corresponding gauge fields $(A_{\mu i},\, B_\mu)$, $i = 1,\, 2,\, 3$. Explain why there are two couplings g, g'. A complex scalar field ϕ is a two component column vector and belongs to a $T = 1/2$ representation of $SU(2)_T$ and also Y is normalised so that its Y-charge is $1/2$. Assume that in the vacuum ϕ has a non zero constant value ϕ_0 chosen so that $\phi_0^\dagger\, \phi_0 = \frac{1}{2}\, v^2$. Show that the gauge group is broken to $U(1)_Q$ with generator Q such that $Q\phi_0 = 0$. Explain why ϕ_0 may be chosen so that $Q = T_3 + Y$.

Define the gauge covariant derivative $D_\mu \phi$ and show that

$$\mathcal{L} \,=\, (D^\mu \phi)^\dagger D_\mu \phi\,,$$

is gauge invariant. When $\phi \to \phi_0$ show that this generates a mass term for the four gauge fields $(A_{\mu i}, B_\mu)$ involving the matrix

$$\tfrac{1}{4}\, v^2 \begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix},$$

whose eigenvalues determine the masses of the gauge fields. What is the form for the matrix representing Q acting on the gauge fields in this basis? Determine the masses and charges of the physical gauge fields.

What representations of $SU(2)_T \times U(1)_Y$ must the electron and its associated neutrino belong to? Show how the gauge invariant Lagrangian can be expressed in terms of a column vector formed by two Dirac fields Ψ and also a single Dirac field ψ. What are the required Y-charges for Ψ and ψ? Explain why in the minimal theory no Dirac or Majorana mass terms, quadratic in the fields, are possible.

Show that if

$$\phi' \,=\, i\, \tau_2\, \phi^*\,,$$

then ϕ' also belongs to a $T = \frac{1}{2}\, SU(2)_T$ representation but has Y-charge $-1/2$. Show that $\phi'^\dagger \Psi$ and $\bar{\Psi}\, \phi'$ are singlets under the gauge group.

Demonstrate how coupling to the scalar ϕ can generate a Dirac mass term for the electron.

[τ_i are the Pauli matrices where $i\, \tau_2\, \tau_i{}^* = -\tau_i\, i\, \tau_2$.]

2

The charge conjugation matrix C is defined so that, if γ^μ are the four dimensional Dirac gamma matrices, $C\,\gamma^{\mu t}C^{-1} = -\gamma^\mu$, where t denotes transpose, and $C^\dagger C = 1$. Show that, for $\gamma_5 = i\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$, $C\,\gamma_5^t\,C^{-1} = \gamma_5$ and $C[\gamma^\mu,\,\gamma^\nu]^t C^{-1} = -[\gamma^\mu,\,\gamma^\nu]$. Why must $C^t = -C$?

If ψ is a Dirac field satisfying

$$\Big(i\,\gamma^\mu(\partial_\mu - ieA_\mu) - m\Big)\psi \,=\, 0\,,$$

find the corresponding equation for $\psi^c = C\bar\psi^t$, where $\bar\psi = \psi^\dagger\gamma^0$ and $\gamma^0\,\gamma^{\mu\dagger}\,\gamma^0 = \gamma^\mu$.

Let ψ satisfy $\gamma_5\psi = \pm\psi$. Show that $\bar\psi\gamma_5 = \mp\bar\psi$ and $\gamma_5\psi^c = \mp\psi^c$. For such an anticommuting chiral fermion field consider the Lagrangian density

$$\mathcal{L} \,=\, \bar\psi\, i\,\gamma^\mu\partial_\mu\psi + \frac{1}{2}\Big(m\,\psi^t\,C^{-1}\psi - m^*\,\bar\psi\,C\,\bar\psi^t\Big)\,.$$

Treating $\psi, \bar\psi$ as independent obtain the equations of motion

$$i\,\gamma^\mu\partial_\mu\psi - m^*\psi^c \,=\, 0\,, \qquad i\,\gamma^\mu\partial_\mu\psi^c - m\,\psi \,=\, 0\,.$$

What is the mass of the field ψ?

With the notation in question 1 show that if

$$\mathcal{L}_G \,=\, G\,(\phi'^\dagger\Psi)^t\,C^{-1}\phi'^\dagger\Psi - G^*\,\bar\Psi\phi'\,C\,(\bar\Psi\phi')^t\,,$$

is added to the lepton Lagrangian then gauge invariance is maintained and when $\phi \to \phi_0$ a non zero mass, which should be calculated, for the neutrino is generated. Why does the presence of $\mathcal{L}_G$ imply that lepton number is no longer conserved?

3

The low energy weak Lagrangian density has the form

$$\mathcal{L}_W = -\frac{G_F}{\sqrt{2}} J^{\alpha\dagger} J_\alpha ,$$

$$J_\alpha = J_\alpha^{\text{leptons}} + J_\alpha^{\text{hadrons}} , \qquad J_\alpha^{\text{leptons}} = \bar{\nu}_e \gamma_\alpha (1-\gamma_5) e + \dots ,$$

where e, ν_e denote the electron, electron neutrino fields. Describe briefly how $\mathcal{L}_W$ gives rise to purely leptonic processes which are not possible just with electromagnetic interactions.

For the decay of a pion, $\pi^-(p) \to e^-(k) + \bar{\nu}_e(q)$, explain why the vector current part of $J_\alpha^{\text{hadrons}}$ does not contribute and that the essential matrix element has the form $\langle 0 | J_\alpha^{\text{hadrons}} | \pi^-(p) \rangle = -i\sqrt{2}\, F_\pi\, p_\alpha$. Calculate the decay rate $\Gamma_{\pi^- \to e^- \bar{\nu}_e}$ and show that it vanishes if $m_e = 0$. What is $\Gamma_{\pi^- \to e^- \bar{\nu}_e} / \Gamma_{\pi^- \to \mu^- \bar{\nu}_\mu}$? Why does this provide a test for the form of $J_\alpha^{\text{leptons}}$? Describe in outline why the corresponding decay rate for the K^- is suppressed by a factor $\sin^2\theta_C$ where θ_C is the Cabbibo angle.

[*The formula for the decay rate of a particle with mass m is*

$$\Gamma = \frac{1}{2m} \sum_X (2\pi)^4 \delta^4(p - p_X) \, |\langle X | \mathcal{L}_I | p \rangle|^2 , \qquad \sum_X = \prod_{\text{momenta}} \int \frac{d^3p}{(2\pi)^3 2p^0} \sum_{\text{spins}} . \Bigg]$$

4

For a hadron H of 4-momentum P^μ, $P^2 = M^2$, represented by the state $|P\rangle$, define

$$\begin{aligned} W^{\mu\nu}(q,P) &= \frac{1}{4\pi}\sum_X (2\pi)^4\delta^4(P-p_X-q)\,\langle P|J^\mu|\,X\rangle\langle X|J^\nu|\,P\rangle \\ &= \Big(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\Big)W_1 + \Big(P^\mu - \frac{P{\cdot}q}{q^2}\,q^\mu\Big)\Big(P^\nu - \frac{P{\cdot}q}{q^2}\,q^\nu\Big)\,W_2\,, \end{aligned}$$

where $J^\mu = \sum_f Q_f \overline{q}_f\,\gamma^\mu q_f$ is the electromagnetic current in terms of quark fields q_f and the state $|X\rangle$ has 4-momentum p_X^μ. If $x = -q^2/(2P{\cdot}q)$ and letting $W_1 = F_1(x,-q^2)$, $P{\cdot}q\,W_2 = F_2(x,-q^2)$, show that as $-q^2\to\infty$ with suitable assumptions,

$$F_1(x,-q^2) \sim \frac{1}{2}\sum_f Q_f^2\Big(f(x)+\overline{f}(x)\Big)\,, \qquad F_2(x,-q^2) \sim x\sum_f Q_f^2\Big(f(x)+\overline{f}(x)\Big)\,.$$

Explain briefly why we may expect

$$\int_0^1 \mathrm{d}x\,\Big(f(x)-\overline{f}(x)\Big) \;=\; N_f\,,$$

where N_f is the number of quarks of type f in the hadron H.

If only u, d quarks are relevant, so that $\overline{q}_f = 0$, and F_2^{proton}, F_2^{neutron} are the functions for H corresponding to a proton and a neutron respectively, what are the values of the integrals

$$\int_0^1 \frac{\mathrm{d}x}{x}\,F_2^{\text{proton}}(x,-q^2)\,, \qquad \int_0^1 \frac{\mathrm{d}x}{x}\,F_2^{\text{neutron}}(x,-q^2)\,,$$

as $-q^2\to\infty$?

$$\Big[\gamma^\mu\gamma^\lambda\gamma^\nu \;=\; g^{\mu\lambda}\gamma^\nu + g^{\nu\lambda}\gamma^\mu - g^{\mu\nu}\gamma^\lambda + i\,\epsilon^{\mu\nu\lambda\kappa}\gamma_\kappa\gamma_5\,.\Big]$$

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