UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 3 June, 2010 9:00 am to 12:00 pm

PAPER 44

STRING THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The worldsheet of a classical relativistic closed string is parameterized by coordinates τ and $\sigma \in [0, 2\pi)$. The embedding of the string into Minkowski space $\mathbf{R}^{3,1}$ is described by the functions $X^\mu(\sigma, \tau)$, with $\mu = 0, 1, 2, 3$. Write down the equations of motion and constraints obeyed by X^μ.

Explain why there are no solutions describing a circular closed string at constant radius, R.

The most general solution to the equations of motion is given by

$$X^\mu(\sigma,\tau) \;=\; X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \qquad \text{where } \sigma^\pm \;=\; \tau \pm \sigma\,.$$

The mode expansion for these fields can be written as

$$X^\mu_L(\sigma^+) \;=\; \frac{1}{2}\,x^\mu + \frac{1}{2}\,\alpha' p^\mu\,\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\,\sum_{n\neq 0}\,\frac{1}{n}\,\tilde{\alpha}^\mu_n\,e^{-in\,\sigma^+}\,,$$

$$X^\mu_R(\sigma^-) \;=\; \frac{1}{2}\,x^\mu + \frac{1}{2}\,\alpha' p^\mu\,\sigma^- + i\sqrt{\frac{\alpha'}{2}}\,\sum_{n\neq 0}\,\frac{1}{n}\,\alpha^\mu_n\,e^{-in\,\sigma^-}\,.$$

Write down the constraints in terms of the modes p^μ and α^ν_n. Use this to provide an expression for the classical mass of the string and explain what is meant by the term "level matching"

Consider now a string moving on $\mathbf{R}^{2,1} \times \mathbf{S}^1$, where the circle $\mathbf{S}^1$ is parameterised by $X^3 \equiv X^3 + 2\pi R$. Explain how the mode expansion changes. Construct a classical solution obeying the constraints which, when projected onto $\mathbf{R}^{2,1}$, looks like a static string of constant radius R.

2

Consider a theory of several free non-interacting scalars X^μ, $\mu = 1, \ldots, D$. The stress-energy tensor is given by

$$T = -\frac{1}{\alpha'} : \partial X^\mu \, \partial X_\mu :$$

Using the fact that ∂X^μ is a primary operator of weight 1 and $: e^{ip\cdot X} :$ is a primary operator of weight $\alpha' p^2/4$, find the conditions for the following operators to be primary

$$\zeta_\mu : \partial X^\mu \, e^{ip\cdot X} : \qquad \text{and} \qquad \zeta_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu \, e^{ip\cdot X} :$$

A free fermion ψ has the operator product expansion

$$\psi(z)\,\psi(w) = -\psi(w)\,\psi(z) = \frac{1}{z-w}\,.$$

For the stress-energy tensor

$$T = -\frac{1}{2} : \psi \partial \psi : \,,$$

show that ψ is primary. Compute its weight. Determine the central charge of this theory.

Consider a closed string propagating in $\mathbf{R}^{3,1}$ together with some number of world-sheet fermionic excitations ψ^i. How many fermions are required for consistency? Describe the spectrum of states with $M^2 \leqslant 0$ associated to vertex operators with either zero or two fermionic insertions.

3

Under an infinitesimal Weyl rescaling and reparametrization, the worldsheet metric $g_{\alpha\beta}$ transforms as

$$\delta\, g_{\alpha\beta} \;=\; 2\,\omega g_{\alpha\beta} + \nabla_\alpha v_\beta + \nabla_\beta v_\alpha\,.$$

Use this to provide a path integral expression for the inverse Faddeev-Popov determinant, $\Delta^{-1}_{FP}[g]$.

Assuming that $\Delta_{FP}[g]$ is gauge invariant, explain the Faddeev-Popov procedure that results in the ghost action,

$$S_{\text{ghost}} \;=\; \int d^2 z \Big(b_{zz}\,\partial_{\bar z}\, c^z + b_{\bar z\bar z}\,\partial_z\, c^{\bar z} \Big)\,.$$

Sketch *in outline* how the presence of the ghosts results in the critical dimension D of the string.

The Veneziano amplitude describing two-to-two tachyon scattering in the open string is given by ratios of Gamma functions

$$\begin{aligned}\mathcal{A} \;\sim\; g_s\big(B(-\alpha' s-1,\,-\alpha' t-1) \;&+\; B(-\alpha' s-1,\,-\alpha' u-1)\\ &+\; B(-\alpha' t-1,\,-\alpha' u-1)\big)\end{aligned}$$

where $B(a,b) \;=\; \Gamma(a)\Gamma(b)/\Gamma(a+b)$. The gamma function $\Gamma(x)$ has simple poles at $x = 0,\,-1,\,-2,\,\ldots$. Determine the open string spectrum from the position of these poles. What information would we obtain by computing the residues of the poles?

The Veneziano amplitude is exponentially suppressed for fixed angle, high-energy scattering. Explain the relevance of this observation.

4

Write down the worldsheet action describing a string coupled to a background spacetime metric $G_{\mu\nu}(X)$, an anti-symmetric tensor $B_{\mu\nu}(X)$ and the dilaton $\Phi(X)$.

What are the local symmetries of the various terms in the action?

Describe how the couplings to $G_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$ are related to the states of the string that arise from quantization in flat space.

Show that the $B_{\mu\nu}$ coupling has a spacetime gauge symmetry. Explain how string perturbation theory arises from the dilaton coupling and the sum over worldsheets of different topologies.

Describe *in outline* how quantization of the two dimensional worldsheet theory leads to equations of motion for fields in $D = 26$ dimensional spacetime.

END OF PAPER