UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2010 1:30 pm to 4:30 pm

PAPER 43

SYMMETRY AND PARTICLE PHYSICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

You may use the property of Pauli matrices, $\sigma_i \, \sigma_j \; = \; \delta_{ij} \, I + i \, \epsilon_{ijk} \, \sigma_k$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the adjoint representation of a Lie algebra of dimension $d(G)$ with basis generators $\{T_a\}$ satisfying $[T_a\,,T_b] = i f_{abc}\,T_c$. Write down the expression for the generators in the adjoint representation.

Let t_a^R be a matrix representation of the generators of the algebra with dimension $d(R)$. Suppose the t_a^R's are normalised so that $\mathrm{tr}(t_a^R\,t_b^R) = C(R)\,\delta_{ab}$. Prove that

$$f_{abc} \;=\; -\frac{i}{C(R)}\,\mathrm{tr}([t_a^R,t_b^R]\,t_c^R)\,.$$

Given that $t_a^R\,t_a^R = C_2(R)\,I$, where $C_2(R)$ is the quadratic Casimir in the representation R, and I is the $d(R)\times d(R)$ unit matrix, show that

$$C(R) \;=\; \frac{d(R)\,C_2(R)}{d(G)}\,.$$

Write the generators of the product of representations, $t_a^{R_1\otimes R_2}$, in terms of the generators of $t_a^{R_1}$ and $t_a^{R_2}$. Given this product, decompose into the sum of generators of irreducible representations,

$$t_a^{R_1\otimes R_2} \;=\; \oplus_i t_a^{R_i}\,,$$

show that

$$(C_2(R_1)+C_2(R_2))\,d(R_1)\,d(R_2) \;=\; \sum_i C_2(R_i)\,d(R_i)\,.$$

Now consider the group $G = SU(N)$. Using tensor notation, or otherwise, show how the representation, $\mathbf{N}\otimes\bar{\mathbf{N}}$ decomposes into the sum of irreducible representations, where $\mathbf{N}$ denotes the fundamental representation and $\bar{\mathbf{N}}$ the anti-fundamental representation.

Use the above equations to deduce the value of the quadratic Casimir in the adjoint representation, $C_2(Ad(SU(N)))$, by considering the special case of $R_1 = \mathbf{N}$ and $R_2 = \bar{\mathbf{N}}$ and making the conventional choice, $C(N) = 1/2$.

2

What is a representation of a Lie algebra and how is it related to a unitary representation of a group?

The angular momentum algebra is expressed in terms of the relations

$$[J_3, J_\pm] \;=\; \pm J_\pm\,, \qquad [J_+, J_-] \;=\; 2J_3\,, \qquad [J_\pm, J_\pm] \;=\; 0\,.$$

A basis of states $\{|m\rangle\}$ are eigenstates of J_3 satisfying $J_3\,|m\rangle \,=\, m\,|m\rangle$. What is meant by a 'highest-weight state' $|m_{max}\rangle$? Prove that states of the form $(J_-)^\ell|m_{max}\rangle$ form a finite-dimensional space if m_{max} satisfies conditions that should be stated. Show that this allows for integer or half-integer spins.

What is the spin-$\frac{1}{2}$ representation of $J_\pm$, J_3?

Explain how a three-dimensional rotation, $R(\theta,\mathbf{n})$, by an angle θ around an axis specified by the unit vector $\mathbf{n}$ is described by the group $SU(2)$. Write down the unitary operator $U(R)$ that generates such rotations acting on basis states. Hence, explain how unitary spin-j representations, $D^{(j)}_{mm'}(R)$, may be defined in terms of representations of the angular momentum algebra.

Consider a rotation parameterised in the form

$$R \;=\; R(\phi,\mathbf{n_3})\,R(\theta,\mathbf{n_2})\,R(\psi,\mathbf{n_3})\,,$$

where $\mathbf{n_2}$, $\mathbf{n_3}$ are unit vectors along the 2, 3 axes and ϕ, ψ, θ are the Euler angles. Write down $U(R)$ for this form of R and hence determine the form of $D^{(j)}_{mm'}(R)$. Show explicitly that $D^{(\frac{1}{2})}_{mm'}(R)$ changes sign under a rotation $\theta = 2\pi$.

Comment on the relation between the groups $SU(2)$ and $SO(3)$.

3

How is a general $SU(3)$ tensor, $T^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l}$ ($\alpha_i, \beta_j \in \{1, 2, 3\}$), defined?

How can it be decomposed into a sum of irreducible representations?

Determine the number of states in the highest dimension irreducible representation obtained from this general tensor.

Quarks are spin-$\frac{1}{2}$ particles described by fermionic (anticommuting) fields that transform in the fundamental representation, denoted **3**, of flavour $SU(3)$ (when the light quarks are taken to have equal masses and the heavy quarks are ignored), as well as the **3** of colour $SU(3)$. There are three quarks in a baryon. Given that baryons are colour singlets (colour 'confinement'), deduce that baryon states in which the quarks have no orbital angular momentum form either an **8** or a **10** of flavour $SU(3)$, but cannot form a singlet state. What are spins of the states in these multiplets.

Antiquarks are also spin-$\frac{1}{2}$ particles that transform in the anti-fundamental representation (the $\bar{\mathbf{3}}$) of both the colour and flavour $SU(3)$ groups. Determine the quantum numbers of the possible $SU(3)$ flavour meson multiplets (colour-singlet states made of one quark and one antiquark).

4

A general complex 2×2 matrix can be written as

$$A(\mathbf{u}) \;=\; u_0\, I + i u_i\, \sigma_i\,,$$

where u_0 and u_i $(i = 1, 2, 3)$ are arbitrary complex numbers, σ_i are the Pauli matrices and I is the two-dimensional unit matrix. What conditions need to be satisfied by (u_0, u_i) for A to be a $SU(2)$ matrix? Show that u_0 may be expressed in terms of $\mathbf{u} = (u_1, u_2, u_3)$ and motivate the statement that the group manifold of $SU(2)$ is S^3.

Use the group property, $A(\mathbf{w}) = A(\mathbf{u})\, A(\mathbf{v})$, to show that the action of an infinitesimal transformation, $A(\mathbf{v}) = I + i v_i\, \sigma_i$, generates an infinitesimal shift in $\mathbf{u}$ given by

$$du_i \;=\; v_j\, \mu_{ji}(\mathbf{u})\,,$$

where $\mu_{ji}(\mathbf{u}) = u_0\, \delta_{ij} + u_k\, \epsilon_{jki}$.

Show that the action of the left-invariant vector fields,

$$T_j(\mathbf{u}) \;=\; i\mu_{ji}(\mathbf{u})\, \frac{\partial}{\partial u_i}\,,$$

on the matrix A is given by $T_j\, A(\mathbf{u}) = -A(\mathbf{u})\, \sigma_j$, and hence that T_i are generators of the algebra,

$$[T_i, T_j] \;=\; -2i\, \epsilon_{ijk}\, T_k\,.$$

The expression $\int d\rho(\mathbf{u})\, f(A(\mathbf{u}))$ denotes the integral of a function $f(A(\mathbf{u}))$ over the $SU(2)$ group manifold, where $d\rho(\mathbf{u})$ is the group invariant measure. Show that

$$d\rho(\mathbf{u}) \;=\; \frac{1}{|u_0|}\, d^3u\,,$$

and hence show that $SU(2)$ is a compact group.

[*You may assume that $T_i(\mathbf{u})$ is a left-invariant vector field on the group manifold.*]

END OF PAPER