UNIVERSITY OF CAMBRIDGE

## MATHEMATICAL TRIPOS Part III

Thursday, 27 May, 2010 9:00 am to 12:00 pm

# PAPER 42

# QUANTUM FIELD THEORY

*Attempt no more than* ***THREE*** *questions.*

*There are* ***FOUR*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

The Klein-Gordon field $\phi(x)$ has mode expansion

$$\phi(x) \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_\mathbf{p}}}\,\left(a_\mathbf{p}e^{-ip.x} + a_\mathbf{p}^\dagger e^{ip.x}\right),$$

where $p = (p^0, \mathbf{p})$.

In the interaction picture explain what is meant by the time ordered product; the normal ordered product; the Feynman propagator. State and prove Wick's theorem and explain how the above quantities are related.

Outline how you would derive the Feynman rules for correlation functions

$$\langle\, 0|T\left(\phi(x_1),\, \phi(x_2)\, \ldots\, \phi(x_n)\right) S|0\,\rangle\ ,$$

where $S$ is the S-matrix in the theory with Lagrange density

$$\mathcal{L} \;=\; \frac{1}{2}\,\partial_\mu\,\phi\,\partial^\mu\,\phi \;-\; \frac{1}{2}\,m^2\,\phi^2 \;-\; \frac{\lambda}{4!}\,\phi^4\ .$$

**2**

The massless Dirac equation is

$$i\,\gamma^\mu\,\partial_\mu\,\psi(x) \;=\; 0$$

where $\psi(x)$ is the spinor wave function and

$$\gamma^0 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\;,\;\gamma^i \;=\; \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where 1 is the $2\times 2$ unit matrix and $\sigma^i$ are the Pauli matrices. Compute $\gamma^5$ where

$$\gamma^5 \;=\; i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3\;.$$

Show that it anticommutes with $\gamma^0$ and $\gamma^i$.

Consider the eigenstates of $\gamma^5$ and show that the Dirac equation becomes two, two component equations.

Show that the left-handed spinor has plane wave solutions of the form

$$\psi(x) \;=\; \lambda(p)\;e^{-ip.x}\;.$$

By considering the condition on the four momentum, $p^\mu$, for such a solution to exist, compute $\lambda(p)$.

Consider spatial rotations about the three momentum axis $\mathbf{p}$ on your solution. What can you infer about the spin states of the particles of the quantised theory?

**3**

The Lagrangian density for a pseudoscalar particle interacting with a fermion is

$$\mathcal{L} \;=\; \frac{1}{2}\,(\partial_\mu\phi)^2 \;-\; \frac{1}{2}\,\mu^2\phi^2 \;+\; \bar{\psi}\,(i\,\gamma^\mu\,\partial_\mu - m)\,\psi \;-\; g\,\phi\,\bar{\psi}\,\gamma^5\,\psi\;.$$

(a) Find the field equation satisfied by $\psi$. By considering the corresponding equation satisfied by $\bar{\psi}$, compute the 4-divergence of the axial current $\bar{\psi}\,\gamma^\mu\,\gamma^5\,\psi$.

(b) State the Feynman rules for the theory given by the Lagrangian density above. Draw the Feynman diagrams for $\psi\psi \to \psi\psi$ and $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering. Write down the scattering amplitudes for these two processes.

**4**

Give an account of the canonical quantisation of the electromagnetic field. You should give the Hamiltonian and the canonical commutation relations. You should discuss gauge invariance and the need to gauge fix. Explain how the quantisation procedure gives rise to photon states.

## END OF PAPER