UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 28 May, 2010 1:30 pm to 3:30 pm

PAPER 41

STATISTICAL FIELD THEORY

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Give an account of the Landau-Ginsberg (LG) theory of phase transitions which should include a discussion of the following points:

(i) The idea of an order parameter;

(ii) The distinction between first-order and continuous phase transitions and how their occurrence is predicted in LG theory;

(iii) The Maxwell construction and the occurence of domains;

(iv) The idea of *critical exponents* and how they may be derived;

(v) The features of a tricritical point, how it occurs in LG theory and a brief description of a 3D phase diagram containing a tricritical point.

State what is meant by the *scaling hypothesis*. For a system described by a single scalar field, show that LG theory predicts that in the neighbourhood of an ordinary continuous phase transition the equilibrium free energy A can be written as

$$A \;=\; a|t|^2\, f_{\substack{< \\ >}}\left(b\,\frac{h}{|t|^{3/2}}\right)\,, \qquad (*)$$

where t is the reduced temperature and h is the magnetic field, i.e. the external field conjugate to the order parameter. Explain the meaning of the subscript $\substack{< \\ >}$ on $f_{\substack{< \\ >}}$.

Use the expression (*) to compute two critical exponents of your choice.

How should (*) be modified to accommodate anomalous scaling behaviour? In this case show that the scaling hypothesis predicts that the scaling relations

$$\alpha + 2\beta + \gamma \;=\; 2\,, \qquad \beta\delta \;=\; \beta + \gamma\,,$$

hold, where the critical exponents $\alpha, \beta, \gamma, \delta$ should be defined.

2

A spin model in D dimensions is defined on a cubic lattice of spacing a with N sites and with spin σ_r on the r-th site. The Hamiltonian is defined in terms of a set of operators $O_i(\{\sigma\})$ by

$$H = \sum_i u_i O_i(\{\sigma\}),$$

where the u_i are coupling constants with $\mathbf{u} = (u_1, u_2, \ldots)$. In particular, H contains the term $-h\sum_r \sigma_r$ where h is the magnetic field. The partition function is given by

$$\mathcal{Z}(\mathbf{u}, C, N) = \sum_\sigma \exp(-\beta H(\mathbf{u}, \sigma) - \beta N C).$$

Define the two-point correlation function $G(\mathbf{r})$ for this model and state how the correlation length ξ parametrizes its behaviour as $r \to \infty$.

Explain how the renormalization group (RG) transformation may be defined in terms of a blocking kernel which, after p iterations, yields a blocked partition function $\mathcal{Z}(\mathbf{u}_p, C_p, N_p)$ that predicts the same large-scale properties for the system as does $\mathcal{Z}(\mathbf{u}, C, N)$. State how a and N rescale in terms of the RG scale factor b.

Derive the RG equation for the free energy $F(\mathbf{u}_p, C_p)$, and explain how it may be expressed in terms of a singular part, $f(\mathbf{u})$, which obeys the RG equation

$$f(\mathbf{u}_0) = b^{-pD} f(\mathbf{u}_p) + \sum_{j=0}^{p-1} b^{-jD} g(\mathbf{u}_j), \qquad p > 0,$$

where the rôle of the function $g(\mathbf{u})$ should be explained.

Explain the idea of a fixed point, *relevant* and *irrelevant* operators, a critical surface and a repulsive trajectory in the context of the RG equations. Sketch some typical RG flows near to a critical surface.

Show how the critical exponents characterizing a continuous phase transition may be derived. In the case where there are two relevant couplings $t = (T - T_C)/T_C$ and h, derive the scaling form for the singular part, F_s, of the free energy:

$$F_s = |t|^{D/\lambda_t} f_\pm\left(\frac{h}{|t|^{\lambda_h/\lambda_t}}\right),$$

where the meanings of λ_t, λ_h and the "$\pm$" subscript on $f_\pm$ should be explained.

The following critical exponents are defined:

$$\begin{array}{rcll} \xi & \sim & |t|^{-\nu} & h = 0\,, \\ C_V & \sim & |t|^{-\alpha} & h = 0\,, \\ M & \sim & |t|^{\beta} & h = 0\,,\ T < T_C\,, \end{array}$$

where C_V is the specific heat at constant volume, and M is the magnetization. Establish the scaling relation $\alpha = 2 - D\nu$.

The Gaussian model in D dimensions for a real scalar field is defined by the Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2}\left(\kappa^{-1}(\nabla\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x})\right) + h\phi(\mathbf{x}) \ ,$$

where κ and h are constants.

By defining a suitable thinning transformation show, for $2 < D < 4$, that the critical exponents α and β are given by

$$\alpha = (4-D)/2 \ , \qquad \beta = (D-2)/4 \ .$$

3

A statistical system in D dimensions and at temperature T is described by a scalar field theory whose effective Hamiltonian is defined by

$$\begin{aligned} H(\phi) &= \int_{\Lambda^{-1}} d^D x\, \mathcal{H}(\Lambda, \phi(\mathbf{x}))\,, \\ \mathcal{H}(\Lambda, \phi(\mathbf{x})) &= \tfrac{1}{2}\,\alpha^{-1}(\nabla\phi(\mathbf{x}))^2 + \tfrac{1}{2}\,m^2(\Lambda, T)\,\phi^2(\mathbf{x}) + \frac{1}{4!}\,g(\Lambda, T)\,\phi^4(\mathbf{x}) + \dots\,, \end{aligned}$$

where Λ is the ultra-violet cut-off and $\mathcal{H}$ is the Hamiltonian density. The magnetic field is set to zero. The partition function is

$$\mathcal{Z} = \int \{d\phi\}\, e^{-H(\phi)}\,.$$

Why do the coupling constants depend on Λ? Why is it reasonable to associate $m(0,T)$ with the correlation length?

By giving an example of a blocking tranformation explain how a Renormalization Group (RG) strategy for successively integrating out high-momentum modes may be applied to this model.

By making suitable assumptions, show how the Landau-Ginsburg theory of phase transitions may be derived using the RG in the context of this model.

In the case of a ϕ^4 scalar field theory, the two-point function $G(\mathbf{x})$ and its Fourier transform $\tilde{G}(\mathbf{p})$ are defined by

$$G(\mathbf{x}) = \langle \phi(0)\,\phi(\mathbf{x})\rangle_c\,, \qquad \tilde{G}(\mathbf{p}) = \int d^D x\, e^{-i\mathbf{p}\cdot\mathbf{x}}\, G(\mathbf{x})\,.$$

State what is meant by the truncated two-point function $\tilde{\Gamma}(\mathbf{p})$.

Explain how, in perturbation theory, $\tilde{\Gamma}(\mathbf{p})$ may be written as

$$\tilde{\Gamma}(\mathbf{p}) = \tilde{G}_0^{-1}(\mathbf{p}) + \delta m^2 + \Sigma(\mathbf{p})\,,$$

where the meaning of each of the terms in this expression should be clearly given. You may quote the rules of perturbation theory without derivation.

Hence show, to one-loop order, that

$$m^2(0,T) = m^2(\Lambda,T) + \frac{g}{2}\int \frac{d^D p}{(2\pi)^D}\,\frac{1}{\mathbf{p}^2 + m^2(0,T)}\,.$$

Show that this result is consistent with the Landau-Ginsburg assumption that $m^2(0,T) \sim (T - T_C)$ only for $D > D_C$, where the value of D_C for an ordinary critical point should be calculated.

Describe briefly how the value of D_C for a tricritical point is calculated and determine its value.

END OF PAPER