UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 3 June, 2010 1:30 pm to 4:30 pm

PAPER 4

TOPICS IN GROUP THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Define what it means for a group to be *nilpotent*.

Prove that a finite group is nilpotent if and only if is is the direct product of its Sylow subgroups.

(ii) For a general finite group G, give two definitions of the Frattini subgroup $\Phi(G)$, one in terms of maximal subgroups of G and the other in terms of generating properties, and show that the two definitions are equivalent. Prove that $\Phi(G)$ is a characteristic nilpotent subgroup of G. If G is a finite p–group, show that $\Phi(G) = G'\, G^p$, where G' is the derived subgroup of G and $G^p = \langle g^p \mid g \in G \rangle$.

(iii) A generating set S for the group G is a *minimal generating set* if no proper subset of S generates G. Show that any two minimal generating sets of the finite p–group G have the same size (Burnside Basis Theorem).

(iv) Show that the symmetric group S_n is generated by an n–cycle and an $(n-1)$–cycle whose product is a transposition.

Show finally that S_n has a generating set consisting of $n-1$ transpositions, but that no set of $n-2$ transpositions can generate S_n.

2

(i) Prove that a primitive permutation group of degree n containing a 3–cycle contains the alternating group A_n.

For each prime $p > 3$, give an example of a primitive group G of degree $n = p+1$ with p dividing $|G|$ but G not containing A_n.

Give an example of an odd prime p and a primitive group G of degree $n = p+2$ with p dividing $|G|$ but G not containing A_n. [*Note that such a G is 3–transitive.*]

(ii) Let T be a group with trivial centre, let $G = T \times T$ and let G act on T by

$$(t_1,\, t_2) \;:\; t \mapsto t_1^{-1}\, t\, t_2 \,.$$

Show that G is faithful and transitive in this action, and that G is primitive on T if and only if T is non–abelian simple.

Give an example of a prime p and a primitive permutation group of degree $p+1$ of order not divisible by p.

3

(i) Show that a maximal subgroup M of a non-abelian simple group G must be the normaliser in G of a characteristically simple group. [*Hint: Consider a minimal normal subgroup of* M.]

Show that A_5 has three conjugacy classes of maximal subgroups, and that they have orders 6, 10 and 12.

Show that $GL_3(2)$ has three conjugacy classes of maximal subgroups, one of order 21 and two of order 24.

(ii) Decribe the Fano projective plane $PG_2(2)$ with 7 points and 7 lines. Interpret the maximal subgroups of $GL_3(2)$ of order 24 in this context, distinguishing the two classes.

Show that if A, B are subgroups of order 24 in different classes, then $A \cap B$ has order 6 or 8, and both possibilities can occur.

Let $\overline{G}$ be the group of order 336 obtained by extending $GL_3(2)$ by the outer automorphism τ which takes any matrix to its inverse transpose. Show that $\overline{G}$ has three conjugacy classes of maximal subgroups (other than $GL_3(2)$), of orders 12, 16 and 42. [*Note that* τ *swaps the two classes in* G *of subgroups of order* 24.]

4

(i) Define the linear groups $GL_n(q)$ and $SL_n(q)$ over the field of q elements, and obtain their orders.

Sketch a proof that the projective group $PSL_n(q)$ is simple for $n > 1$, except when $n = 2$ and $q \leqslant 3$.

(ii) Indicate the changes needed in (i) when dealing with the symplectic groups $Sp_{2m}(q)$.

(iii) Consider the action of $GL_n(q)$ on the set X_k of k–dimensional subspaces of $V_n(q)$, with $1 \leqslant k \leqslant n/2$. Describe the stabiliser P_k of an element of X_k.

What is the rank of $GL_n(q)$ on X_k?

(iv) The graph Γ_k has as its vertices the elements of X_k, with two vertices A, B joined by an edge if $\dim(A \cap B) = k - 1$. Show that the distance d in Γ_k satisfies

$$d(A, B) \;=\; k - \dim(A \cap B)\,.$$

Deduce that $G = GL_n(q)$ is distance–transitive on Γ_k: if $d(A_1, A_2) = d(B_1, B_2)$, then $A_1\, g = B_1$ and $A_2\, g = B_2$ for some $g \in G$.

5

Write an essay on finite 2–transitive groups. Include many examples.

END OF PAPER