UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 28 May, 2010 1:30 pm to 4:30 pm

PAPER 39

ADVANCED FINANCIAL MODELS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let r be a process defined by a constant r_0 and

$$r_{t+1} = \beta r_t + \xi_{t+1}$$

for $t \geqslant 0$, where β is a given parameter and $\xi_1, \xi_2, \ldots$ is a sequence of independent and identically distributed random variables. Suppose the function K defined by $K(\theta) = \log \mathbb{E}[e^{\theta \xi_1}]$ is everywhere finite.

(a) Show that

$$r_t = \beta^t r_0 + \sum_{s=1}^{t} \beta^{t-s} \xi_s$$

for all $t \geqslant 0$.

(b) Consider a market with an asset whose price B is given by $B_0 = 1$ and

$$B_{t+1} = e^{r_{t+1}} B_t$$

for $t \geqslant 0$. For $0 \leqslant t \leqslant T$, introduce a family of zero coupon bonds with prices

$$P(t,T) = \mathbb{E}\left[\frac{B_t}{B_T} | \mathcal{F}_t\right]$$

where $\mathcal{F}_t$ is the sigma-field generated by $\xi_1, \ldots, \xi_t$.

Show that

$$P(t,T) = e^{Q(t,T) r_t + R(t,T)}$$

for non-random functions Q and R to be determined in terms of the parameter β and the function K.

(c) Let $p : \mathbb{Z}_+ \to (0,\infty)$ be a given function with $p(0) = 1$. By replacing the original model with

$$r_{t+1} = \beta r_t + \alpha_{t+1} + \xi_{t+1},$$

show that there are constants $\alpha_1, \alpha_2, \ldots$ such that $P(0,T) = p(T)$ for all $T \geqslant 0$.

2

Consider a two-asset model where the price of asset 0 is $B_t = 1$ for all $t \geqslant 0$ and the price $(S_t)_{t \geqslant 0}$ of asset 1 is a non-negative martingale. Introduce to this market a family of contingent claims with payout $\sqrt{S_T}$ indexed by the maturity date $T \geqslant 0$. Suppose that the prices of these claims are given by the formula

$$C(t,T) \;=\; \mathbb{E}[\sqrt{S_T}|\mathcal{F}_t]\,.$$

for $0 \leqslant t \leqslant T$.

(a) Use Jensen's inequality to show that for fixed $t \geqslant 0$, the function $T \mapsto C(t,T)$ is decreasing almost surely.

(b) Suppose for the moment that the dynamics of S are given by

$$dS_t \;=\; S_t \sigma_0 dW_t$$

where W is a Brownian motion and σ_0 is a non-negative constant. Show that the prices of the claims in this case can be written as $C(t,T) = F(t,T,S_t,\sigma_0)$ for a non-random function F to be determined.

(c) Now suppose that S is given by

$$dS_t \;=\; S_t \sigma_t dW_t$$

for a bounded, predictable process σ. Let $\Sigma(t,T)$ be the unique $\mathcal{F}_t$-measurable non-negative random variable such that

$$\mathbb{E}[\sqrt{S_T}|\mathcal{F}_t] \;=\; F(t,T,S_t,\Sigma(t,T)) \text{ almost surely,}$$

where F is defined in (b).

Suppose $a \leqslant \sigma_t \leqslant b$ a.s. for all $t \geqslant 0$ for non-negative constants a and b. If σ and W are independent, show

$$a \leqslant \Sigma(t,T) \leqslant b \text{ a.s for all } 0 \leqslant t < T\,.$$

(d) By finding a suitable change of measure, show that the above inequality also holds even when σ and W are not assumed to be independent.

3

Let $Y = (Y_t)_{t \in \{0, \dots, T\}}$ be a given integrable process adapted to the filtration $(\mathcal{F}_t)_{t \in \{0, \dots, T\}}$. The Snell envelope of Y is the process U defined inductively by $U_T = Y_T$ and

$$U_t = \max\{Y_t, \mathbb{E}[U_{t+1} | \mathcal{F}_t]\}$$

for $0 \leqslant t \leqslant T - 1$.

(a) Show that U is a supermartingale. Show that U is a martingale if Y is a submartingale.

(b) Let $\xi_1, \dots, \xi_T$ be an i.i.d. sequence of random variables, let $X_0 = 0$,

$$X_t = \xi_1 + \dots + \xi_t ,$$

and let $\mathcal{F}_t$ be the sigma-field generated by $\xi_1, \dots, \xi_t$. Fix a measurable function $f : \mathbb{R} \to \mathbb{R}$ and let $Y_t = f(X_t)$. Suppose that Y_t is integrable for each $t \geqslant 0$, and let U be the Snell envelope of Y. Show that there exists a deterministic function V such that $U_t = V(t, X_t)$.

(c) Prove by induction that if the function f is convex then the functions $V(t, \cdot)$ are convex for each $0 \leqslant t \leqslant T$.

[Recall that a function $\phi : \mathbb{R} \to \mathbb{R}$ is called convex if

$$\phi[\theta x + (1 - \theta) y] \leqslant \theta \phi(x) + (1 - \theta) \phi(y)$$

for all $x, y \in \mathbb{R}$ and $0 < \theta < 1$.]

4

Consider a $d+1$ asset, continuous-time model where asset 0 is a riskless numéraire. Assume that the asset prices are modelled by a $(d+1)$-dimensional Itô process (B, S).

(a) What is an arbitrage? What is an equivalent martingale measure? Prove that the existence of an equivalent martingale measure implies the absence of arbitrage. You may use standard facts from stochastic calculus without proof as long as they are clearly stated.

(b) Now, suppose that for each share of asset i held during the infinitesimal interval $(t-dt, t]$, the agent receives $D_t^i dt$ units of money at time t where the d-dimensional dividend process D is continuous and adapted. Explain why the agent's wealth process X satisfies the two equations

$$\begin{aligned} X_t &= \phi_t B_t + \pi_t \cdot S_t \\ dX_t &= \phi_t dB_t + \pi_t \cdot D_t dt + \pi_t \cdot dS_t \,. \end{aligned}$$

(c) Show that there is no arbitrage if there exists an equivalent measure $\mathbb{Q}$ such that the process Y defined by

$$Y_t = \frac{S_t}{B_t} + \int_0^t \frac{D_s}{B_s}\, dt$$

is a $\mathbb{Q}$-martingale.

5

Consider a two asset, continuous time model (B, S) where

$$\begin{aligned} dB_t &= B_t r dt \\ dS_t &= S_t(\mu dt + \sigma dW_t) \end{aligned}$$

for constants r, μ and $\sigma > 0$ and Brownian motion W.

(a) Introduce a claim with payout $\xi_T = g(S_T)$ where $T > 0$ is the maturity date and g is a bounded smooth function. Suppose the price of the claim is given by $\xi_t = V(t, S_t)$ for a deterministic, bounded function V which satisfies the Black–Scholes PDE. Show that the payout ξ_T can be replicated by an admissible strategy $(\pi_t)_{t\in[0,T]}$.

(b) Show that V can be expressed in the form

$$V(t, S_t) = e^{-r(T-t)}\mathbb{E}[g(S_t e^{a(t)+b(t)Z})]$$

where Z is a standard normal random variable, and a and b are functions to determine.

(c) Suppose g has a bounded derivative g'. Show that if g is an increasing function, then ξ_T can be replicated by a strategy satisfying $\pi_t \geqslant 0$ almost surely for all $t \geqslant 0$.

6

Consider a $d+1$ asset, one-period model. Asset 0 is riskless with $B_0 = 1$ and $B_1 = 1+r$ for a constant r. Assets 1, ... , d have time-0 prices $S_0 \in \mathbb{R}^d$ and time-1 prices S_1 with mean $\mu = \mathbb{E}(S_1)$ and $d \times d$ covariance matrix $V = \mathbb{E}[(S_1 - \mu)(S_1 - \mu)^{\mathrm{T}}]$. Assume V is invertible.

(a) To this market, introduce a contingent claim with time-1 payout ξ_1 where ξ_1 is bounded. Find the portfolio $(\phi^*, \pi^*) \in \mathbb{R}^{1+d}$ which minimizes the expected squared hedging error

$$\mathbb{E}[(\xi_1 - X_1)^2]$$

where $X_1 = \phi B_1 + \pi \cdot S_1$.

(b) Show that there exists a random variable ρ^* such that the optimal initial capital $X_0^* = \phi^* B_0 + \pi^* \cdot S_0$ can be expressed in the form

$$X_0^* = \mathbb{E}[\rho^* \xi_1].$$

(c) Verify the equality $\mathbb{E}[\rho^* \bar{S}_1] = \bar{S}_0$ where $\bar{S} = (B, S)$. Prove that

$$\mathbb{E}[\rho^2] \geqslant \mathbb{E}[(\rho^*)^2]$$

for every random variable ρ such that $\mathbb{E}[\rho \bar{S}_1] = \bar{S}_0$.

(d) Now suppose $S_1 \sim N_d(\mu, V)$ has the normal distribution, and that $\xi_1 = g(S_1)$ where g has bounded gradient ∇g. Show that

$$X_0^* = \frac{1}{1+r}\mathbb{E}[g(S_1)] + \left(S_0 - \frac{1}{1+r}\mu\right) \cdot \mathbb{E}[\nabla g(S_1)].$$

[You may use without proof the Gaussian integration by parts formula $\mathbb{E}[Zf(Z)] = \mathbb{E}[\nabla f(Z)]$, where $Z \sim N_d(0, I)$ and I is the $d \times d$ identity matrix.]

END OF PAPER