UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 4 June, 2010 9:00 am to 11:00 am

PAPER 38

ACTUARIAL STATISTICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $S = X_1 + \ldots + X_N$ (if $N = 0$ then $S = 0$) be a random sum with 'steps' $X_1, X_2, \ldots$ given by independent and identically distributed (iid) positive random variables, and where the number N of summands is independent of the steps. Show that S has moment generating function $M_S(u) = G_N[M_{X_1}(u)]$ where G_N is the probability generating function of N and M_{X_1} is the moment generating function of X_1.

A portfolio consists of n independent policies where for policy i, $i = 1, \ldots, n$, the number of claims during an accounting period is N_i independent of the sizes of claims which are iid with distribution function F_i.

(a) Suppose that N_i has a Poisson distribution with mean λ_i. Show that the distribution of the total amount T claimed on the portfolio during a typical accounting period has a compound Poisson distribution. In the case where $F_1 = F_2 = \ldots = F_n = F$ and $F(x) = 1 - e^{-x}$, $x \geqslant 0$, show that the distribution function of T can be written in the form

$$F_T(x) = a + (1-a)\tilde{F}_T(x), \qquad x \geqslant 0,$$

where a is a constant in $(0,1)$ which you should specify, and where $\tilde{F}_T$ has density

$$\tilde{f}_T(x) = \sum_{k=1}^{\infty} \frac{\lambda^k}{(e^{\lambda}-1)k!} \frac{x^{k-1}e^{-x}}{(k-1)!}.$$

(b) Now suppose that, for $i = 1, \ldots, n$, $\mathbb{P}(N_i = k) = q^k p$, $k = 0, 1, \ldots$, where $q = 1-p$ and $0 < p < 1$, and that $F_1 = F_2 = \ldots = F_n = F$ where F is as in (a). Show that T is distributed as a random sum, and state the distribution of the steps and of the number of summands for this random sum. When $n = 2$ show that

$$F_T(x) = b + (1-b)\check{F}_T(x), \qquad x \geqslant 0,$$

where b is a constant between 0 and 1 and $\check{F}_T$ is a distribution function. State the value of b and write down a density for $\check{F}_T$.

2

Consider the total amount claimed in a year on a particular risk where the number of claims N has $\mathbb{P}(N = n) = p_n$, $n = 0, 1, 2, \ldots$, and where claims are independent and identically distributed random variables $X_1, X_2, \ldots$ independent of N. Suppose that

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1} , \qquad n = 1, 2, \ldots$$

for some constants a and b. Suppose also that the claim sizes are positive and discrete with $\mathbb{P}(X_1 = j) = f_j$, $j = 1, 2, \ldots$. Let S be the total amount claimed in a year, and let $g_r = \mathbb{P}(S = r)$, $r = 0, 1, \ldots$. Show the following recursion for $\{g_r\}_{r=0}^{\infty}$:

$$g_0 = p_0 , \quad g_r = \sum_{j=1}^{r} \left(a + \frac{bj}{r}\right) f_j \, g_{r-j} \quad r \geqslant 1 .$$

Now assume that N has a Poisson distribution with mean λ. Write down the recursion for $\{g_r\}_{r=0}^{\infty}$ for this case, and derive a recursion for $\{\mathbb{E}(S^k)\}_{k=1}^{\infty}$. Use this recursion to find $\mathbb{E}(S)$, $\mathrm{var}(S)$ and $\mathbb{E}\big((S - \mathbb{E}(S))^3\big)$ in terms of λ and the moments of X_1.

3

In the classical risk model, let $M_X(r)$ be the moment generating function of the claim sizes, let λ be the claim arrival rate, let the premium loading factor be $\theta > 0$ (so that the premium rate is $c = (1 + \theta)\lambda\mu$ where μ is the expected claim size), and let $\psi(u)$ be the probability of ruin with initial capital $u \geqslant 0$. You are given that there is a unique positive solution R of the equation $M_X(r) - 1 = (1 + \theta)\mu r$. Prove that $\psi(u) \leqslant e^{-Ru}$ for $u \geqslant 0$ (the Lundberg inequality).

(a) Suppose that the claim sizes are exponentially distributed with mean 1. Find R.

(b) Suppose that the claim sizes have density

$$f_X(x) = e^{-2x} + \frac{1}{3} e^{-2x/3} , \qquad x > 0 .$$

Write down the expected claim size. Show that R is the smaller root of

$$3(1 + \theta) r^2 - (8\theta + 5) r + 4\theta = 0 .$$

If mistakenly the claims are assumed to be exponentially distributed with the same mean, compare the resulting upper bounds on the probability of ruin given by the Lundberg inequality.

4

Let X_i be the amount claimed on a risk in year i, $i = 1, 2, \ldots$, and suppose that, given θ, the X_i's are independent and identically distributed with density

$$f(x \mid \theta) = \frac{\theta^k e^{-\theta/x}}{x^{k+1}(k-1)!}, \qquad x > 0,$$

where $k > 2$ is a known positive integer. Suppose that the prior density of θ is

$$\pi(\theta) = \frac{\lambda^\alpha \theta^{\alpha-1} e^{-\lambda\theta}}{(\alpha-1)!}, \qquad \theta > 0,$$

where α is a known positive integer and $\lambda > 0$ is known. Suppose that $X_1, \ldots, X_n$ are observed and consider $\mu(\theta) = \mathbb{E}(X_{n+1} \mid \theta)$. Find $c_0, c_1, \ldots, c_n$ in terms of known quantities such that $\mathbb{E}\left[\left(\mu(\theta) - c_0 - \sum_{i=1}^n c_i X_i\right)^2\right]$ is minimised. Define what is meant by a credibility estimate, and show that $c_0 + \sum_{i=1}^n c_i X_i$ can be written in the form of a credibility estimate. Discuss the effect on the credibility factor as

(a) k increases while α and λ remain fixed;

(b) the prior variance of θ decreases while the prior mean of θ and k remain fixed.

Find the Bayesian estimate of $\mathbb{E}(X_{n+1} \mid \theta)$ under quadratic loss. State whether or not this can be written in the form of a credibility estimate.

[Hint:

(i) You may assume without proof that if Y has density

$$f(x) = \frac{\theta^k e^{-\theta/x}}{x^{k+1}(k-1)!}, \qquad x > 0,$$

where $k \in \{3, 4, \ldots\}$ and $\theta > 0$, then

$$\mathbb{E}(Y) = \frac{\theta}{(k-1)}, \quad \mathbb{E}(Y^2) = \frac{\theta^2}{(k-1)(k-2)}, \quad \textit{and} \quad \mathrm{var}(Y) = \frac{\theta^2}{(k-1)^2(k-2)}.$$

(ii) Let random variables U, V and W be such that U and V have finite second moments. Then

$$\mathrm{cov}(U, V) = \mathbb{E}\big(\mathrm{cov}(U, V \mid W)\big) + \mathrm{cov}\big(\mathbb{E}(U \mid W), \mathbb{E}(V \mid W)\big).\,]$$

END OF PAPER