UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2010 1:30 pm to 4:30 pm

PAPER 35

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the optimization problem

$$\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & h(x) = b \\ \text{over} & x \in X\,, \end{array}$$

where $X \subset \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Define the Lagrangian function for this problem and then state and prove the Lagrangian Sufficiency Theorem. Define the function ϕ by

$$\phi(b) \;=\; \inf_{x \in X} \{f(x) : h(x) = b\}\,.$$

Define the Strong Lagrangian property and show that the following are equivalent:

(a) there exists a non-vertical supporting hyperplane to ϕ at b;

(b) the problem is Strong Lagrangian.

Minimize $f = \sum v_i\, x_i^{-1}$ in $x \geqslant 0$ subject to $\sum a_i\, x_i \leqslant b$ where $a_i, v_i > 0$ for all i and $b > 0$. [In this example f is the variance of an estimate derived from a stratified sample survey subject to a cost constraint: x_i is the size of the sample for the i^{th} stratum, the a_i and v_i are measures of sampling cost and of variability for this stratum.]

Check that the change in the minimal variance f for a small change δb in available resources is $\lambda \delta b$ where λ is the Lagrange multiplier.

2

(a) A Company wishes to maximize its profit by solving the optimization problem

$$\begin{array}{ll} \text{Maximize} & 3\,x_1 + 4\,x_2 \\ \text{subject to} & x_1 + 2\,x_2 \leqslant 6 \\ & 2\,x_1 + x_2 \leqslant 6 \\ \text{over} & x_1 \geqslant 0\,, \quad x_2 \geqslant 0\,. \end{array}$$

Use the simplex algorithm in tableau form to solve this linear problem.

(b) Subsequently the Company realises it has forgotten to include an additional constraint

$$4x_1 + 4x_2 \leqslant 15 \tag{1}$$

Use the dual simplex algorithm on the simplex tableau derived from part (a) to solve the linear program

$$\begin{array}{ll} \text{Maximize} & 3\,x_1 + 4\,x_2 \\ \text{subject to} & x_1 + 2\,x_2 \leqslant 6 \\ & 2\,x_1 + x_2 \leqslant 6 \\ & 4\,x_1 + 4\,x_2 \leqslant 15 \\ \text{over} & x_1 \geqslant 0\,, \quad x_2 \geqslant 0\,. \end{array}$$

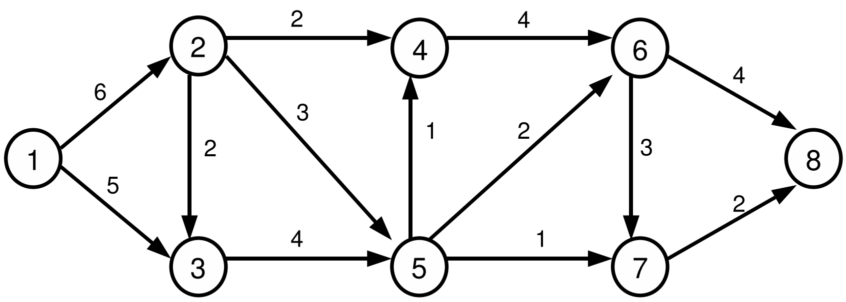
(c) Later the Company forms an agreement to sell slack in the constraint (1). Justifying any alteration made, alter the simplex tableau derived from part (b) and use the simplex algorithm to solve linear program

$$\begin{array}{ll} \text{Maximize} & 3\,x_1 + 4\,x_2 + y \\ \text{subject to} & x_1 + 2\,x_2 \leqslant 6 \\ & 2\,x_1 + x_2 \leqslant 6 \\ & 4\,x_1 + 4\,x_2 + y = 15 \\ \text{over} & x_1 \geqslant 0\,, \quad x_2 \geqslant 0\,, \quad y \geqslant 0\,. \end{array}$$

3

(a) State and prove the Max-Flow Min-Cut Theorem.

(b) Use the Ford-Fulkerson Algorithm to calculate a maximum flow between a source at node 1 and a destination at node 8 in the following network.



Here the number by each arc represents the capacity of that arc. What is the min-cut of this network?

(c) Starting from the given feasible solution, minimize the cost of flows in the transportation problem given by the following tableau.

$\boxed{5}$	7 $\boxed{4}$	5 $\boxed{8}$	12
5 $\boxed{3}$	$\boxed{7}$	$\boxed{2}$	5
$\boxed{9}$	7 $\boxed{0}$	$\boxed{3}$	7
4 $\boxed{8}$	$\boxed{10}$	8 $\boxed{5}$	12
9	14	13	

[Note: In this tableau the circled numbers indicate an initial feasible set of non-zero flows, the numbers in the squares are the costs, the numbers to the right of the tableau are supplies and the numbers below are demands.]

4

Consider the Boolean formula with N clauses

$$(x_{11} \vee x_{12} \vee \ldots \vee x_{1M_1}) \wedge (x_{21} \vee x_{22} \vee \ldots \vee x_{2M_2}) \wedge \ldots \wedge (x_{N1} \vee \ldots V x_{NM_N}) \quad (1)$$

where $x_{ij} \in \{X_1, \ldots, X_K\} \cup \{\bar{X}_1, \ldots, \bar{X}_K\}$. Here $\wedge$ means "AND" and $\vee$ means "OR" and $\bar{X}$ means "NOT X".

The SAT problem considers the assignment of variables, $X_i \in \{\text{true}, \ \text{false}\}$, $i = 1, 2, \ldots, K$ such that (1) is true.

The MAX-SAT problem considers the assignment of variables such that the maximum number of clauses in (1) are true.

Express the SAT problem as an integer linear program.

Express the MAX-SAT problem as an integer linear program.

Consider the following approximation algorithm for MAX-SAT.

Greedy: Pick the variable $z \in \{X_1, \ldots, X_K\} \cup \{\bar{X}_1, \ldots, \bar{X}_K\}$ that occurs in the largest number of clauses in (1). Set z true and $\bar{z}$ false. This reduces formula (1) to an expression on $K-1$ variables. Repeat until no variables remain.

Show that Greedy is a $\frac{1}{2}$-approximation of MAX-SAT.

[*Recall:* Algorithm H with solution α_H is an ϵ-approximation to a maximization problem with optimal solution α^* if for all problem instances,

$$\alpha_H \geqslant (1-\epsilon)\alpha^*.]$$

5

For a coalitional game, explain what is meant by the *characteristic function*, an *imputation* and the *core*.

(a) A group of n miners have discovered large and equal sized lumps of gold. Two miners can carry one lump, so that the payoff of a coalition S is

$$v(S) \;=\; \begin{cases} |S|/2 & \text{if } |S| \text{ is even} \\ (|S|-1)/2 & \text{if } |S| \text{ is odd}\,. \end{cases}$$

Determine the core in the cases where n is even and where n is odd.

Determine the core if it require three miners to carry one lump.

(b) A pair of shoes consists of a left shoe and a right shoe, and can be sold for £10. Consider a coalitional game with $a+b$ players: a of the players have one left shoe each, and b of the players have one right shoe each. Determine the core for each pair of positive integers (a,b).

6

Let X be a convex set of strategies. Recall that a strategy $x^* \in X$ is an evolutionary stable strategy (ESS) if for every $y \in X$, $y \neq x^*$ then

$$e(x^*, \bar{x}) \; > \; e(y, \bar{x})$$

where $\bar{x} = (1-\epsilon)x^* + \epsilon y$ for sufficiently small $\epsilon > 0$. Briefly discuss the interpretation of $e(.\,,.)$, x^* and y in this definition.

Show that a strategy x^* is an ESS if and only if for every $y \in X$, $y \neq x^*$

$$e(x^*, x^*) \; \geqslant \; e(y, x^*)$$

and if $e(x^*, x^*) = e(y, x^*)$ then

$$e(x^*, y) \; > \; e(y, y)\,.$$

Suppose that a strategy $x \in X$ is a mixture $(p, 1-p)$ of the two pure strategies "Hawk" $= (1,0)$ and "Dove" $= (0,1)$ and that for the pure strategies the pay off matrix is

$$\begin{array}{cc} & \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{array}{c} \text{Hawk} \\ \text{Dove} \end{array} & \begin{pmatrix} \frac{1}{2}(V-D) & V \\ 0 & \frac{1}{2}V \end{pmatrix} \end{array}$$

Find an ESS when

(i) $V > D$,

(ii) $V = D$,

(iii) $V < D$,

justifying your answer in each case.

END OF PAPER