UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 3 June, 2010 9:00 am to 11:00 am

PAPER 30

STOCHASTIC NETWORKS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a) Consider a clock. The clockhand can point to states $\{0, 1, 2, ..., M-1\}$. The clockhand can make clockwise transitions $0 \mapsto 1, 1 \mapsto 2, ..., M-2 \mapsto M-1$ and $M-1 \mapsto 0$. The clockhand can also make anti-clockwise transitions $0 \mapsto M-1, M-1 \mapsto M-2, ..., 2 \mapsto 1$ and $1 \mapsto 0$. The clockhand moves as a Markov chain. The rates of this Markov chain are so that, if the clockhand is at state m then the clockhand moves clockwise at rate ν and the clockhand moves anti-clockwise at rate λ_m. We assume $\prod_{m=0}^{M-1} \lambda_m = \nu^M$. By considering its time reversal, calculate the equilibrium distribution of this Markov chain. In equilibrium what is the process that describes the points in time where the clock makes anti-clockwise transitions?

b) We now connect the clock from part a) with a single server queue. Every time the clockhand makes an anti-clockwise transition a customer arrives at the single server queue. Customers arriving at the queue have an independent exponentially distributed mean 1 service requirement. The server at the queue serves at rate μ where $\mu > \nu$. What is the equilibrium distribution of the Markov chain formed by the clockhand and the queue?

c) Suppose we modify the queueing network from part b), so that now the clockhand can only turn clockwise when a customer departs the single server queue. Assuming this queueing network starts at the state where the clockhand is at 0 and the single server queue is empty, what is the equilibrium distribution of this Markov chain?

2

a) Define the states and transitions made by multi-class single server queue. Write down the transition rates of a multi-class single server queue. Stating the method you use, calculate equilibrium distribution for this process.

b) Similarly, define the states and transitions made by a network of multi-class single server queues. Write down the transition rates and traffic equations for a network of multi-class single server queues. Calculate the equilibrium distribution for this process.

[In parts a) and b) it is assumed that customers have a service requirement that is exponentially distributed.]

3

a) Let $E(\nu, C)$ denote Erlang's formula for the loss probability of an Erlang link with $C > 0$ circuits and where calls arrive as a Poisson process of rate $\nu > 0$. Prove that

$$E(N\nu, NC) \to \max\left\{0, 1 - \frac{C}{\nu}\right\} \quad \text{as} \quad N \to \infty.$$

b) Now consider a Loss Network with fixed routing, with routes $r \in R$, links $j \in J$ and where link j comprises of C_j circuits. Let A_{jr} equal 1 when link j is used by route r and otherwise let A_{jr} equal 0. Calls arrive on each route r as a Poisson process of rate ν_r. Write down the Erlang Fixed Point Approximation for a Loss Network with fixed routing. Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & \sum_{r \in R} \nu_r e^{-\sum_{j \in J} y_j A_{jr}} + \sum_{j \in J} \int_0^{y_j} U(z, C_j) dz \\ \text{over} \quad & y_j \geqslant 0, \quad j \in J, \end{aligned} \tag{1}$$

where the function U is defined to satisfy

$$U(-\log(1 - E(\nu, C)), C) = \nu(1 - E(\nu, C)).$$

Use optimization problem (1) to prove uniqueness of the Erlang Fixed Point Approximation for a Loss Network with fixed routing. You may assume the function $U(z, C)$ is strictly increasing in z.

c) By part a) or by considering an optimization problem similar to (1) discuss why there need not be a unique solution to the fixed point equation

$$B = E(\nu[1 + 2B(1 - B)], C).$$

4

a) Consider a Loss Network with fixed routing, with routes $r \in R$, links $j \in J$ and where link j comprises of C_j circuits. Let A_{jr} equal 1 when link j is used by route r and otherwise let A_{jr} equal 0. Calls arrive on each route r as a Poisson process of rate ν_r. The stationary distribution of such a Loss Network is of the form

$$\pi(n) = \frac{1}{G(C)} \prod_{r \in R} \frac{\nu_r^{n_r}}{n_r!}, \qquad n \in S(C),$$

where $S(C) = \{n \in \mathbb{Z}_+^R : \sum_{r \in R} A_{jr} n_r \leqslant C_j,\ j \in J\}$ and $G(C)$ is a normalizing constant.

Consider the optimization problem

$$\begin{aligned} \text{maximize} \quad & \sum_{r \in R} x_r \log \nu_r + x_r - x_r \log x_r \\ \text{subject to} \quad & \sum_{r: j \in r} x_r \leqslant C_j, \quad j \in J \qquad\qquad (\text{NETWORK}(\nu)) \\ \text{over} \quad & x_r \geqslant 0, \quad r \in R. \end{aligned}$$

Describe how this optimization problem can be used to approximate the stationary distribution of a loss network with fixed routing.

b) Consider the optimization problems

$$\begin{aligned} \text{maximize} \quad & \sum_{r \in R} U_r(x_r) \\ \text{subject to} \quad & \sum_{r: j \in r} x_r \leqslant C_j, \quad j \in J \qquad\qquad (\text{SYSTEM}) \\ \text{over} \quad & x_r \geqslant 0, \quad r \in R; \end{aligned}$$

and for each $r \in R$,

$$\begin{aligned} \text{maximize} \quad & U_r(\nu_r e^{-y_r}) - \nu_r y_r e^{-y_r} \qquad\qquad (\text{USER}(U_r,y_r)) \\ \text{over} \quad & \nu_r \geqslant 0; \end{aligned}$$

and the equalities

$$x_r = \nu_r e^{-y_r}, \qquad r \in R. \tag{1}$$

Here we assume, for each $r \in R$, U_r is strictly concave, increasing, continuously differentiable on $(0, \infty)$ and $U_r'(0) = \infty$.

Show that there exists $x^* = (x_r^* : r \in R)$, $y^* = (y_r^* : r \in R)$ and $\nu^* = (\nu_r^* : r \in R)$ that are such that (1) is satisfied, such that ν_r^* solves the USER(U_r,y_r^*) problem for each $r \in R$, such that x^* solves the NETWORK(ν^*) optimization problem and such that x^* solves the SYSTEM optimization problem.

END OF PAPER