UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2010 9:00 am to 12:00 pm

PAPER 3

COMMUTATIVE ALGEBRA

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Suppose that A is a ring. Define an Artinian A-module. What does it mean to say that A is Artinian?

Prove carefully that if A is an Artinian ring and M is a finitely generated A-module then M is an Artinian A-module.

Give an example of an Artinian ring A and a non-Artinian A-module M.

Show that if M is an Artinian A-module and f is an injective endomorphism of the A-module M, then f is surjective.

2

Suppose that A is a ring. Define the Zariski topology on $\mathrm{Spec}(A)$. Show that if A is a finitely generated $\mathbb{C}$-algebra then there is a 1-1 correspondance between radical ideals of A and closed subsets of $\mathrm{maxSpec}(A)$ equipped with the subspace topology.

Describe this correspondance explicitly for $A = \mathbb{C}[x]$.

3

Suppose that A is a subring of an integral domain B. What does it mean to say that B is integral over A?

Define the Krull dimension of a ring. Show that if B is integral over A then the two rings have the same Krull dimension.

Suppose that K is an algebraic field extension of $\mathbb{Q}$ and $\mathcal{O}$ is its ring of integers. What is the Krull dimension of $\mathcal{O}$? Justify your answer.

4

Suppose that A is an integral domain. Define $\mathrm{Pic}(A)$ the Picard group of A and $\mathrm{Cart}(A)$ the group of Cartier divisors of A. Carefully show that there is a natural exact sequence of abelian groups

$$1 \to A^{\times} \to Q(A)^{\times} \to \mathrm{Cart}(A) \to \mathrm{Pic}(A) \to 0\,.$$

5

Suppose that A is a ring and M is an A-module. Let $M^* := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

Explain how to give M^* a natural structure as an A-module and show that the following are equivalent:

1. M is flat;
2. M^* is injective;
3. $I \otimes_A M \cong IM$ for each ideal I of A;
4. $\mathrm{Tor}_1^A(A/I, M) = 0$ for each ideal I of A.

You should prove any results about flatness and injectivity that you use.

6

Suppose A and B are rings. Discuss the construction and properties of the right derived functors R_iF of a left exact contravariant functor

$$F\colon A\text{--mod} \to B\text{--mod}.$$

Illustrate your discussion with the functor $M \mapsto \mathrm{Hom}_A(A/Aa, M)$ in the case that $B = A/Aa$ for $a \in A$ not a zero divisor.

END OF PAPER