UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 28 May, 2010 9:00 am to 12:00 am

PAPER 29

STOCHASTIC CALCULUS AND APPLICATIONS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Fix a measurable function $\psi : \mathbb{R} \to [0,\infty)$. For a cadlag function $X : [0,\infty) \to \mathbb{R}$, define

$$V_t^{n,\psi} = \sum_{k=0}^{\lceil 2^n t \rceil - 1} \psi\left(X_{(k+1)2^{-n}} - X_{k2^{-n}}\right) .$$

(a) Assume that $\psi(u) = |u|,\ u \in \mathbb{R}$. Show that $\lim_{n\to\infty} V_t^{n,\psi}$ exists in $[0,\infty]$ for all $t \geqslant 0$. What is the limit V_t called? Show that V_t is cadlag. Give without proof an expression for $\Delta V_t = V_t - V_{t-}$ in terms of X.

(b) Assume that X is a continuous local martingale in some filtered probability space satisfying the usual conditions, and for $u \in \mathbb{R}$ let $\psi(u) = |u|^p$, where $p > 2$. Show that $\lim_{n\to\infty} V_t^{n,\psi} = 0$, uniformly on compacts in probability (u.c.p).

(c) Now assume that X is a continuous local martingale with $X_0 = 0$, and that $\psi(u) = |u|^p$ for $u \in \mathbb{R}$, where $1 < p < 2$. Assume that almost surely, $\limsup_{n\to\infty} V_t^{n,\psi} < \infty$ for some $t \geqslant 0$. Show that X is indistinguishable from 0 on $[0,t]$.

(d) Let $\psi(u) = u^2$, and let X be a continuous martingale satisfying $\sup_{t\geqslant 0} \mathbb{E}(X_t^2) < \infty$. Without assuming anything about the quadratic variation of martingales, show that for all $t \geqslant 0$, $\mathbb{E}(V_t^{n,\psi})$ remains bounded as $n \to \infty$.

2

Let X be a standard Gaussian random variable in $\mathbb{R}^d$, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a function of class C^2 on $\mathbb{R}^d$, such that for some $K > 0$, $\|\nabla f(x)\| \leqslant K$ for all $x \in \mathbb{R}^d$, where $\| \cdot \|$ stands for the Euclidean norm in $\mathbb{R}^d$. The goal of this problem is to show that

$$\mathbb{P}(|f(X) - \mu| > r) \leqslant 4e^{-r^2/(2K)}, \tag{1}$$

for all $r > 0$, where $\mu = \mathbb{E}(f(X))$.

(a) Let $(X_t, t \geqslant 0)$ be a d-dimensional Brownian motion started at 0, and let $\mathcal{F}_t = \sigma(X_s, 0 \leqslant s \leqslant t)$. For $0 \leqslant t \leqslant 1$, let $M_t = \mathbb{E}(f(X_1)|\mathcal{F}_t)$. Show that

$$M_t = P_{1-t}f(X_t), \quad a.s.,$$

where for any $s \geqslant 0$, $P_s f(x) = \mathbb{E}_x[f(X_s)]$ is the semi-group of Brownian motion in $\mathbb{R}^d$, and $\mathbb{E}_x$ indicates that the process starts at $x \in \mathbb{R}^d$.

(b) Using Itô's formula, show that

$$dM_t = \sum_{i=1}^{d} P_{1-t} \frac{\partial f}{\partial x_i}(X_t) dX_t^i,$$

where X^i denotes the i^{th} coordinate process of X. [In applying Itô's formula, you are not required to verify that the function is C^2. It may simplify your calculations to observe that M is a martingale.]

(c) Let $[M]$ denote the quadratic variation of M. Show that $[M]_1 \leqslant K^2$. Then, using the Dubins-Schwarz theorem (which you should state), show that

$$\mathbb{P}(|M_1 - M_0| > r) \leqslant 2\mathbb{P}\left(\sup_{0 \leqslant s \leqslant K^2} B_s > r \right),$$

where B is a one-dimensional Brownian motion.

(d) Using the reflection principle or otherwise, deduce (1). [You may admit the following result without proof: let Z be a standard Gaussian random variable, then $\mathbb{P}(Z > r) \leqslant e^{-r^2/2}$ for all $r \geqslant 0$.]

[Non-examinable: the conclusion (1) can be extended to arbitrary Lipschitz functions, through what is called Rademacher's theorem].

3

For $x > 0$, let $\mathbb{W}_x$ denote the Wiener measure started at x, i.e., the probability measure on the space of continuous trajectories $\Omega = C([0,\infty),\mathbb{R})$ under which the canonical process $(X_t, t \geqslant 0)$ is a (one-dimensional) Brownian motion started at x. We equip Ω with its Borel σ-algebra $\mathcal{F}$ and the canonical filtration $(\mathcal{F}_t, t \geqslant 0)$. For $y \in \mathbb{R}$, let $T_y = \inf\{t \geqslant 0 : X_t = y\}$, and let $\tau = T_M \wedge T_\varepsilon$, where $0 < \varepsilon < x < M$ are fixed for the moment. Define also $M_t = \int_0^{t\wedge T_0} dX_s/X_s$, and let $Z_t = \exp(M_t - \frac{1}{2}[M]_t)$.

(a) State Girsanov's theorem. Define a probability measure $\mathbb{Q}^{M,\varepsilon}$ on $(\Omega, \mathcal{F})$, by saying $\mathbb{Q}^{M,\varepsilon}(A) = \mathbb{E}_{\mathbb{W}_x}(\mathbf{1}_A Z_\tau)$ for all $A \in \mathcal{F}$. Show that, under $\mathbb{Q}^{M,\varepsilon}$, $X_t = x + B_t + \int_0^t \frac{ds}{X_s}$, for all $t \leqslant \tau$, where B is a $\mathbb{Q}^{M,\varepsilon}$-Brownian motion.

[You may assume without proof that Z^τ is uniformly integrable.]

(b) Let $(W_t, t \geqslant 0)$ be a three-dimensional Brownian motion, started from $\bar{x} = (x, 0, 0)$, and let $R_t = \|W_t\|$. R is called a (three-dimensional) *Bessel process* started at x. Show, using Itô's formula, that $dR_t = dB_t + \frac{1}{R_t}dt$, where B is one-dimensional Brownian motion, and deduce that the law of $(R_{t\wedge\tau}, t \geqslant 0)$, is the same as $(X_{t\wedge\tau}, t \geqslant 0)$ under $\mathbb{Q}^{M,\varepsilon}$.

(c) Using Itô's formula, show that for $t \leqslant \tau$, $d(\log Z_t) = d\log(X_t)$, and hence deduce $Z_t = X_t/x$ for $t \leqslant \tau$, $\mathbb{W}_x$ almost surely.

(d) By letting $\varepsilon \to 0$, show that the law $\mathbb{Q}^M$ of $(R_{t\wedge T_M}, t \geqslant 0)$, where R is a Bessel process started at x, is absolutely continuous with respect to $\mathbb{W}_x$, with density

$$\frac{d\mathbb{Q}^M}{d\mathbb{W}_x} = \mathbf{1}_{\{T_M < T_0\}} \frac{M}{x} .$$

Conclude that $\mathbb{Q}^M$ is the law of a Brownian motion conditioned to hit M before hitting 0, and stopped at M. [Thus, informally, a Bessel process is a Brownian motion conditioned to hit $+\infty$ before hitting 0.]

4

Let $(B_t, t \geqslant 0)$ be a Brownian motion in $\mathbb{R}^2$.

(a) State and prove the theorem showing that the law of B is conformally invariant. [You may assume other results of the course if they are clearly stated.]

(b) Assume that B starts at $B_0 = (1,0)$. For $\theta \in (-\pi, \pi)$ let $\Delta_\theta = \{z \in \mathbb{C}, z = re^{i\theta}$ for some $r \geqslant 0\}$ be the semi-infinite line passing through the origin and of angle θ with respect to the real axis, and let $S_\theta = \inf\{t \geqslant 0 : B_t \in \Delta_\theta\}$, when we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Fix $0 < \alpha, \beta < \pi$. Using conformal invariance and the holomorphic function $z \mapsto e^z$, compute $\mathbb{P}(S_\beta < S_{-\alpha})$.

(c) Let $R_t = \|B_t\|$ be the radial part of B, and let θ_t be a continuous determination of the argument of B_t such that $\theta_0 = 0$, hence $B_t = R_t e^{i\theta_t}$. [$(\theta_t, t \geqslant 0)$ is only defined on the event that B never hits 0, which holds almost surely]. For $r > 0$, let $T_r = \inf\{t \geqslant 0 : R_t = e^{-r}\}$. Show that $(\theta_{T_r}, r \geqslant 0)$ is a Lévy process, i.e., has independent and stationary increments.

5

(a) Let $(H_t, t \geqslant 0)$ be a left-continuous and locally bounded process which is adapted to a filtration $(\mathcal{F}_t)$ satisfying the usual conditions. Let X be a continuous semimartingale. Show that

$$\sum_{k=0}^{\lfloor 2^n t \rfloor - 1} H_{k2^{-n}}(X_{(k+1)2^{-n}} - X_{k2^{-n}}) \to \int_0^t H_s dX_s,$$

as $n \to \infty$, uniformly on compacts in probability.

(b) Define the *covariation* between two continuous semimartingales X and Y. State the integration by parts formula, and prove it using (a). [You may use without proof any result from the course about the quadratic variation of a continuous semimartingale, but you may not use Itô's formula.]

6

(a) Let X be a continuous adapted process defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with values in $\mathbb{R}$, and let $a(x)$, $b(x)$ be two measurable real functions. Explain what it means to say that X solves the *martingale problem* $\mathbf{M}(a, b)$ associated with a and b. What does it mean to say X is a diffusion generated by L, where $Lf(x) = (1/2)a(x)f''(x) + b(x)f'(x)$, for sufficiently smooth functions f?

Show that if X solves $\mathbf{M}(a, b)$ and $a(x) = \sigma^2(x) > 0$ for all $x \in \mathbb{R}$ for some measurable function σ, then X solves a suitable stochastic differential equation. [You may assume Lévy's characterisation of Brownian motion, provided you state it clearly.] Summarise the relationships between diffusion processes and solutions of martingale problems.

(b) Let $n \geqslant 1$, and $a > 0$. Consider a population of bacteria that evolves in discrete time according to the following Markovian dynamics. If the population size is currently $k \geqslant 1$, it increases by one with probability $p_k = \alpha k/n$, decreases by one with probability $q_k = \beta k^2/n^2$, where $\alpha, \beta > 0$, and otherwise stays constant. The evolution stops if $p_k + q_k > 1$. Initially the population size is $\lfloor \alpha n/\beta \rfloor$. We assume that $2\alpha^2 < \beta$.

Let $(Y_0^n, Y_1^n, \ldots)$ denote the evolution of the total population size, and let $\tilde{Y}^n$ is the linear interpolation of Y^n. Define the rescaled process by

$$X_t^n = \frac{\tilde{Y}^n_{\lfloor nt \rfloor} - n\frac{\alpha}{\beta}}{\sqrt{n}}\,.$$

Show that $(X^n_{t\wedge\tau}, 0 \leqslant t \leqslant 1) \to (X_t, 0 \leqslant t \leqslant 1)$ weakly as $n \to \infty$, where X is the solution of a certain stochastic differential equation which you should determine, and τ is the time at which the evolution of X^n stops.

[State carefully the result you are applying.]

Deduce that $\mathbb{P}(\tau \leqslant 1) \to 0$ as $n \to \infty$ (a brief justification suffices).

END OF PAPER