UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 7 June, 2010 1:30 pm to 4:30 pm

PAPER 28

ADVANCED PROBABILITY

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Biased Random Walks.

Let $(X_n)_{n \geqslant 1}$ be a sequence of iid random variables such that $\mathbb{P}(X_1 = 1) = p$ and $\mathbb{P}(X_1 = -1) = 1 - p =: q$, and define $S_n = X_1 + \cdots + X_n$. We shall write $\mathcal{F}_n$ for $\sigma(X_1, \ldots, X_n)$; all martingales will be with respect to the filtration $(\mathcal{F}_n)_{n \geqslant 1}$.

a) Define for $n \geqslant 1$ the random variable $Z_n = \left(\frac{q}{p}\right)^{S_n}$. Prove that the process $(Z_n)_{n \geqslant 1}$ is a martingale.

b)

(i) State Doob's maximal inequality. Use it to prove that we have for any $k \geqslant 1$

$$\mathbb{P}(\sup_n S_n \geqslant k) \leqslant \left(\frac{p}{q}\right)^k$$

(ii) If $q > p$,

$$\mathbb{E}\,[\sup_n S_n] \leqslant \frac{p}{q-p}\,.$$

c) Suppose that $q > p$.

(i) Find the limit of Z_n as n goes to infinity.

(ii) Given $k \geqslant 1$, set $H_k = \inf\{n \geqslant 1 \; ; S_n \geqslant k\}$. Prove that

$$\lim_{n \to +\infty} Z_{n \wedge H_k} = e^{\lambda k}\, \mathbf{1}_{H_k < \infty}$$

and compute $\mathbb{P}(H_k < \infty)$.

(iii) Hence prove that $\sup_n S_n$ has a geometric distribution of parameter $1 - \frac{p}{q}$.

2 Empirical Distributions

a) Let $(X_n)_{n\geqslant 1}$ be a sequence of iid random variables with uniform law on $[0,1]$. Define the empirical distribution function as

$$F_n(t) \;=\; \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{X_k\leqslant t}\,, \qquad t\,\in\,[0,1]\,.$$

(i) Prove that $\sup_{t\in F}\big|F_n(t)-t\big|$ converges almost-surely to 0 as n goes to infinity for any finite family F of elements of $[0,1]$.

(ii) Prove that $\sup_{x\in[0,1]}\big|F_n(x)-x\big|$ converges almost-surely to 0 as n goes to infinity.

(iii) Let μ be any (Borel) probability measure on $\mathbb{R}$ and $(Y_k)_{k\geqslant 1}$ an iid sequence of random variables with common distribution μ; set $G(x)=\mu\big((-\infty,x]\big)$; Denote by G_n the empirical distribution function of the sequence $(Y_k)_{k\geqslant 1}$

$$G_n(x) \;=\; \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{Y_k\leqslant x}\,, \qquad x\,\in\,\mathbb{R}\,.$$

Using the result established in **(ii)**, prove that $\sup_{x\in\mathbb{R}}\big|G_n(x)-G(x)\big|$ converges almost-surely to 0 as n goes to infinity.

b)

(i) State the definition of a Gaussian process indexed by an arbitrary set T. Why do the mean and covariance functions determine uniquely the law of the process? You may assume any standard properties of Gaussian random variables.

(ii) A Brownian bridge is a centered Gaussian process $(Z_t)_{t\in[0,1]}$ with covariance function $\mathbb{E}[Z_t\,Z_s]=s(1-t)$, for $0\leqslant s<t\leqslant 1$. Show, by specifying a suitable construction of a Brownian bridge, that the sample paths may be taken to be almost-surely continuous.

(iii) In this part we come back to the setting of part **a**. Set

$$Z_n(t) \;=\; \sqrt{n}\,\big(F_n(t)-t\big)\,, \qquad t\,\in\,[0,1]\,.$$

Prove that the finite dimensional laws of Z_n converge to the finite dimensional laws of a Brownian bridge.

3 Symmetric Sequences of Random Variables (1)

This question can be answered independently of question **4**

Denote by $(\Omega, \mathcal{F})$ the product space $\mathbb{R}^{\mathbb{N}}$ equipped with its product σ-algebra. Denote by $(X_n)_{n \geqslant 0}$ the coordinate process on Ω; for a generic element ω of Ω we have $\omega = \big(X_n(\omega)\big)_{n \geqslant 0}$. Set $\mathcal{E}_n = \sigma(X_0, \ldots, X_n)$, $\mathcal{G}_n = \sigma(X_{n+1}, X_{n+2}, \ldots)$ and $\mathcal{G} = \bigcap_{n \geqslant 0} \mathcal{G}_n$.

Denote also by $\mathfrak{S}$ the group of permutations of $\mathbb{N}$ which fix all but finitely many elements of $\mathbb{N}$.

A **probability** $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is said to be **symmetric** if the process $(X_{\sigma(n)})_{n \geqslant 0}$ has the same law as $(X_n)_{n \geqslant 0}$ under $\mathbb{P}$ for any permutation $\sigma \in \mathfrak{S}$.

Last, denote by $\mathcal{S}$ the σ-algebra of Ω made up of events $A \in \mathcal{F}$ such that $\omega = \big(X_n(\omega)\big)_{n \geqslant 0} \in A$ iff $\big(X_{\sigma(n)}(\omega)\big)_{n \geqslant 0} \in A$, for any permutation $\sigma \in \mathfrak{S}$.

a) Let A be an event in $\mathcal{S}$, and write F for $\mathbf{1}_A$. Why do the following convergence results hold in $\mathbb{L}^1$?

$$F = \lim_{n \to +\infty} \mathbb{E}\left[F \middle| \mathcal{E}_n\right], \qquad \mathbb{E}\left[F \middle| \mathcal{G}\right] = \lim_{n \to +\infty} \mathbb{E}\left[F \middle| \mathcal{G}_n\right].$$

b) Suppose now that the probability $\mathbb{P}$ is symmetric. Given $\epsilon > 0$, pick n large enough to have

$$\left\| F - \mathbb{E}\left[F \middle| \mathcal{E}_n\right] \right\|_1 \leqslant \epsilon, \qquad \text{and} \qquad \left\| \mathbb{E}\left[F \middle| \mathcal{G}\right] - \mathbb{E}\left[F \middle| \mathcal{G}_n\right] \right\|_1 \leqslant \epsilon.$$

Prove that $\left\| F - \mathbb{E}\big[F \big| \sigma(X_{n+1}, \ldots, X_{2n+1})\big] \right\|_1 \leqslant \epsilon$ and conclude that the event A coincides $\mathbb{P}$-almost-surely with an event of the σ-algebra $\mathcal{G}$. [Hint: prove that the operator $Z \in \mathbb{L}^1 \rightsquigarrow \mathbb{E}[Z|\mathcal{G}_n] \in \mathbb{L}^1$ is a contraction.]

4 Symmetric Sequences of Random Variables (2)

This question can be answered independently of question **3**. *We use the same notations as in question* **3**.

The aim of this problem is to prove that *if $\mathbb{P}$ is a symmetric probability on the product space $(\Omega, \mathcal{F})$ then the random variables $X_0, X_1, \ldots$ are independent conditionally on the σ-algebra $\mathcal{S}$* . This means that, for any $p \geqslant 0$ and any Borel sets $A_0, \ldots, A_p$ of $\mathbb{R}$, we have $\mathbb{P}$-almost-surely

$$\mathbb{E}\left[\prod_{i=0}^{p} \mathbf{1}_{A_i}(X_i)\Big|\mathcal{S}\right] = \prod_{i=0}^{p} \mathbb{E}\left[\mathbf{1}_{A_i}(X_i)\big|\mathcal{S}\right] . \tag{1}$$

Given $m \geqslant 1$, denote by $\mathcal{S}_m$ the σ-algebra on Ω made up of events $A \in \mathcal{F}$ such that $\omega = \big(X_n(\omega)\big)_{n\geqslant 0} \in A$ iff $\big(X_{\sigma(n)}(\omega)\big)_{n\geqslant 0} \in A$, for any permutation σ fixing all indices greater than m. These σ-algebras decrease to $\mathcal{S}$. Given $k \in \{0, \ldots, p\}$, set

$$S_m^k = \mathbf{1}_{A_k}(X_0) + \cdots + \mathbf{1}_{A_k}(X_m) .$$

a) Fix k in $\{0, \ldots, p\}$. Using the same reasoning as in the backward martingale proof of the strong law of large numbers given in the lectures, prove that $\dfrac{S_m^k}{m+1}$ converges $\mathbb{P}$-almost-surely to $\mathbb{E}\left[\mathbf{1}_{A_k}(X_k)\big|\mathcal{S}\right]$ as $m \to +\infty$. Prove, as a consequence, that we have $\mathbb{P}$-almost-surely

$$\prod_{k=0}^{p} \mathbb{E}\left[\mathbf{1}_{A_k}(X_k)\big|\mathcal{S}\right] = \lim_{m\to+\infty} \frac{1}{(m+1)^{p+1}} \sum_{0\leqslant \ell_0,\ldots,\ell_p\leqslant m} \mathbf{1}_{A_0}(X_{\ell_0}) \ldots \mathbf{1}_{A_p}(X_{\ell_p}) .$$

b) For $m \geqslant p$ prove that

$$\mathbb{E}\left[\prod_{k=0}^{p} \mathbf{1}_{A_k}(X_k)\big|\mathcal{S}_m\right] = \frac{1}{(m+1)m \ldots (m+1-p+1)} \sum \mathbf{1}_{A_0}(X_{\ell_0}) \ldots \mathbf{1}_{A_p}(X_{\ell_p}) .$$

where the sum is over all sets of distinct integers $\{\ell_0, \ldots, \ell_p\}$ in $\{0, \ldots, m\}$.

c) Combining parts **a)** and **b)** prove identity (1).

5 Brownian Motion and Holomorphic Functions

Let U be an open set of the complex plane $\mathbb{C}$. Recall that a complex valued function $f : U \to \mathbb{C}$ is ***holomorphic*** if its differential, as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}$, at any point of U. Write $f = \Re f + i\Im f$. The composition of two holomorphic functions is a holomorphic function. The logarithm function is a well-defined holomorphic function on the half-plane $\{(x, y); y \geqslant 1\}$.

a) Prove that $\triangle\big(\Re f\big) = \triangle\big(\Im f\big) = 0$.

b)

(i) Use the optional stopping theorem to prove that $\Re f$ and $\Im f$ cannot have a local maximum in U without being constant.

(ii) Let $g : U \to \mathbb{C}$ be a holomorphic function. Considering $f = \log g$ on a well-chosen set, deduce from **(i)** that $|g|$ cannot have a local maximum in U without being constant.

c) Without using **b)**, and using the fact that the reciprocal of a non-vanishing holomorphic function is holomorphic, and the properties of the 2-dimensional Brownian motion, prove the fundamental theorem of algebra: *Any non-constant polynomial* $P : \mathbb{C} \to \mathbb{C}$ *has* 0 *in its range.*

d) Give another proof of that fact using **b)**.

6 Hitting Time Process

Let $X = (X_t)_{t \geqslant 0}$ be a real-valued Brownian motion defined on some probability space.

a) Given $0 \leqslant \alpha < \beta < \infty$, prove that $\mathbb{P}\big((X_t)_{t \in [\alpha, \beta]} \text{ is non-decreasing}\big) = 0$. Deduce that, almost-surely, there exists no interval of positive length on which X is non-decreasing.

For $a \geqslant 0$, set $H_a = \inf\{s \geqslant 0 \,;\, X_s = a\}$ and $S_a = \inf\{s \geqslant 0 \,;\, X_s > a\}$.

b) Given $a \geqslant 0$, show that H_a and S_a are almost-surely equal. Yet, show by an illustration that the two processes $H = (H_a)_{a \geqslant 0}$ and $S = (S_a)_{a \geqslant 0}$ are not almost-surely equal.

c) State the definition of a Lévy process, and prove that S is a Lévy process. Why is H not a Lévy process?

d) Prove that S is almost-surely nowhere continuous.

END OF PAPER