UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 2 June, 2010 9:00 am to 11:00 am

PAPER 27

SCHRAMM–LOEWNER EVOLUTIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question you may use without proof any facts you wish concerning conformal isomorphisms and complex Brownian motion, provided these are clearly stated.

(a) What does it mean to say that K is a compact $\mathbb{H}$-hull? Explain how to define the half-plane capacity $\text{hcap}(K)$. Show that for any $r \in (0, \infty)$ we have

$$\text{hcap}(rK) = r^2 \,\text{hcap}(K).$$

Show further that

$$\text{hcap}(K) \leqslant \text{rad}(K)^2$$

where $\text{rad}(K)$ is the radius of the smallest ball centred on the real axis which contains K.

(b) Define for a compact $\mathbb{H}$-hull K

$$c(K) = \lim_{y \to \infty} y \mathbb{P}_{iy}(B_T \in K)$$

where B is a complex Brownian motion starting from iy and

$$T = \inf\{t \geqslant 0 : B_t \notin \mathbb{H} \setminus K\}.$$

Show that $c(K)$ is well defined and that for any $r \in (0, \infty)$ we have

$$c(rK) = rc(K).$$

(c) Show that, for all compact $\mathbb{H}$-hulls K we have

$$\text{hcap}(K) \leqslant \text{rad}(K) c(K).$$

2 Let $(K_t)_{t \geqslant 0}$ be an increasing family of compact $\mathbb{H}$-hulls such that $\text{hcap}(K_t) = 2t$ for all t. What does it mean to say that $(K_t)_{t \geqslant 0}$ has the *local growth property*?

Define the *Loewner transform* $(\xi_t)_{t \geqslant 0}$ of $(K_t)_{t \geqslant 0}$ and show that $(\xi_t)_{t \geqslant 0}$ is continuous. [You may assume that, for some $C < \infty$, for all compact $\mathbb{H}$-hulls K and all $z \in \mathbb{H} \setminus K$, we have $|z - g_K(z)| \leqslant C \,\text{rad}(K)$.]

State Loewner's theorem for such families of compact $\mathbb{H}$-hulls.

Let $\kappa \in [0, \infty)$. What is meant by saying that a continuous process $(\gamma_t)_{t \geqslant 0}$ is $SLE(\kappa)$ in $(\mathbb{H}, 0, \infty)$?

Show that $SLE(\kappa)$ is scale-invariant.

3 Fix $x, y \in (0, \infty)$ with $x < y$ and fix $a \in (1/4, 1/2)$. Denote by $(X_t : t < \sigma)$ and $(Y_t : t < \tau)$ the unique maximal solutions to the coupled Bessel stochastic differential equations

$$dX_t = dB_t + \frac{a}{X_t} dt, \quad dY_t = dB_t + \frac{a}{Y_t} dt, \quad X_0 = x, \quad Y_0 = y.$$

Here B is a real Brownian motion. You may assume that, almost surely, the lifetimes σ, τ are both finite, that $X_t \to 0$ as $t \to \sigma$ and $Y_t \to 0$ as $t \to \tau$, and that, almost surely

$$A := \int_0^\tau \frac{dt}{Y_t^2} = \infty.$$

Show that, for a suitable function χ, the process $N_t = \chi(\frac{Y_t - X_t}{Y_t})$ is a local martingale up to σ, and deduce that

$$\mathbb{P}(\sigma < \tau) = \phi\left(\frac{y-x}{y}\right)$$

where ϕ is given by

$$\phi(\theta) \propto \int_0^\theta \frac{du}{u^{2-4a}(1-u)^{2a}}, \quad \phi(1) = 1.$$

Explain the relevance of this calculation to the study of SLE.

4 Let Φ denote the conformal automorphism of the upper half-plane $\mathbb{H}$ such that $\Phi(-1) = \infty$, $\Phi(0) = 0$ and $\Phi(\infty) = 1$. Let γ be an $SLE(6)$ in $(\mathbb{H}, 0, \infty)$ and let γ' be an $SLE(6)$ in $(\mathbb{H}, 0, 1)$. Set $\gamma_t^* = \Phi(\gamma_t)$. Write $(g_t : t \geqslant 0)$ for the Loewner flow associated with γ and set $\xi_t = g_t(\gamma_t)$. Set

$$S = \inf\{t \geqslant 0 : \gamma_t \in (-\infty, -1]\}.$$

For $t < S$, write K_t^* for the compact $\mathbb{H}$-hull generated by $\gamma^*[0,t]$ and write g_t^* for the conformal isomorphism $\mathbb{H} \setminus K_t^* \to \mathbb{H}$ such that $g_t^*(z) - z \to 0$ as $z \to \infty$. Set $\xi_t^* = g_t^*(\gamma_t^*)$. Consider for $t < S$ the conformal automorphism of $\mathbb{H}$ given by $\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}$. You may assume that Φ_t is C^1 in t and extends analytically to $(g_t(-1), \infty)$. You may assume also the identities

$$\operatorname{hcap}(\gamma^*[0,t]) = 2\int_0^t \Phi_s'(\xi_s)^2 ds, \quad \dot{\Phi}_t(\xi_t) = -3\Phi_t''(\xi_t).$$

(a) Show that $(\xi_t^* : t < S)$ is a local martingale.

(b) Define

$$T = \inf\{t \geqslant 0 : \gamma_t \in [1, \infty)\}, \quad T' = \inf\{t \geqslant 0 : \gamma_t' \in [1, \infty)\}.$$

Show that $(\gamma_t : t < T)$ and $(\gamma_t' : t < T')$, considered up to reparametrization, have the same distribution.

(c) Comment on the behaviour of γ and γ' after they hit $[1, \infty)$. Where does your argument for (b) break down if one seeks to apply it beyond the time that γ and γ' hit $[1, \infty)$?

END OF PAPER