UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 31 May, 2010 1:30 pm to 3:30 pm

PAPER 26

ALGEBRAIC NUMBER THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the Kummer–Dedekind theorem. Determine which primes ramify in $\mathbb{Q}(\zeta_{80})/\mathbb{Q}$, where ζ_n denotes a primitive n-th root of 1.

[*You may assume that the ring of integers of* $\mathbb{Q}(\zeta_n)$ *is* $\mathbb{Z}[\zeta_n]$.]

2

(i) Let F/K be a Galois extension of number fields. Let $\mathfrak{p}$ be a prime of K with residue field $k_{\mathfrak{p}}$, and $\mathfrak{q}$ a prime of F above $\mathfrak{p}$ with residue field $k_{\mathfrak{q}}$. Prove that the natural map from the decomposition group of $\mathfrak{q}$ to $\mathrm{Gal}(k_{\mathfrak{q}}/k_{\mathfrak{p}})$ is surjective.

Now let $F = \mathbb{Q}(\zeta_3, \sqrt[3]{2})$, where ζ_3 denotes a primitive cube root of 1.

(ii) Prove that no prime of F has absolute residue degree 6.

(ii) The prime 7 decomposes in $\mathbb{Q}(\zeta_3)$ as $\mathfrak{p}_1\mathfrak{p}_2$, where $\mathfrak{p}_1 = (\zeta_3 + 3)$ and $\mathfrak{p}_2 = (\zeta_3^2 + 3)$. Determine the Frobenius element of $\mathfrak{p}_1$ in $F/\mathbb{Q}(\zeta_3)$.

3

Let $F = \mathbb{Q}(\sqrt{-2}, \sqrt{-3})$, and let ρ be the regular representation of $\mathrm{Gal}(F/\mathbb{Q}) \simeq C_2 \times C_2$, *i.e.* the direct sum of its four 1-dimensional representations. Compute the first ten coefficients $a_1, \ldots, a_{10}$ of its Artin L-series $L(\rho, s) = \sum_{n\geqslant 1} a_n n^{-s}$.

4

State and prove Chebotarev's density theorem. Prove that for a monic irreducible polynomial $f(X)$ with integer coefficients, there are infinitely many primes p such that $f(X) \mod p$ has no roots in $\mathbb{F}_p$.

[*You may assume that Artin L-functions have meromorphic continuation to $\mathbb{C}$, analytic on $\Re(s) > 1$, that the Riemann ζ-function $\zeta(s)$ has a simple pole at $s = 1$, and that $L(\rho, s)$ is analytic and non-zero at $s = 1$ for non-trivial irreducible representations ρ.*]

END OF PAPER